Relaxed Partitions

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1. Introduction

Recall that a partition of a positive integer n is a finite nonincreasing sequence of positive integers λ_1, $\lambda_1 \dots \lambda_k$ whose sum is equal to n. A lot of beautiful theories and conjectures have been developed in this area and this field is blooming.

In this paper, we are going to take a road less traveled and study an object which we call "relaxed partitions", or more specifically, r-partitions with r to be specified. Unlike the traditional partitions where we require $\lambda_i - \lambda_{i+1} \geq 0$, for r-partitions we require $\lambda_i - \lambda_{i+1} \geq r$ where r can be negative. For example, $(2,3,1,1)$ is a (-1)-partition of 7.

Just as with traditional partitions, there are many questions one could ask about r-partitions. Perhaps one of the first questions to ask is: "for a fixed r, how many r-partitions of the integer n do we have?" Using an easy recurrence relation, we used Maple to program a procedure which we called $\mathbf{NPr(n,r)}$, and it returns the number of r-partitions of n for any given n and r. But can we find a generating function for a given r? The answer turned out to be yes! And there is a nice single-sum formula for it. By typing in a sequence of entries produced by $\mathbf{NPr(n,-1)}$ into the **OEIS**, we found that its generating function seemed to be the reciprocal of

$$\sum_{k=0}^{\infty} \frac{(-1)^k q^{k^2}}{\prod_{i=1}^{k}(1-q^i)} \quad .$$

Recognizing this is the generating function for the "weighted" (with weight $(-1)^k$) number of (traditional) partitions of integer n into parts with difference at least 2, Doron Zeilberger provided an short and elegant bijective proof for it (interested reader can look up the proof in the comments of the sequence **A003116** in **OEIS**). This proof can be directly generalized to general r, and the generating function for the number of r-partitions of n is the reciprocal of

$$\sum_{k=0}^{\infty} \frac{(-1)^k q^{k(2+(1-r)(k-1))/2}}{\prod_{i=1}^{k}(1-q^i)} \quad ,$$

which is the "weighted" number of partitions of integer n into parts with difference at least $(-r+1)$.

In this paper, we will focus on restricted r-partitions such that the first part and the number of the parts are fixed. Let $a_r(M,N,n)$ be the number of r-partitions of n with the first part equal to M and exactly N parts. To go with the notation in our Maple package **rPar**, let $F(M,N,r,q)$ be the generating function for $a_r(M,N,n)$. Using a simple recurrence relation $F(M,N,r,q)$ satisfies, we were able to program it in Maple and happily used Maple to conjecture (**and prove!**) the closed form for the case $q=1$ (i.e., the total number of r-partitions with the first part equal to M and exactly N parts). It is:

$$F(M,N,r,1) = \frac{(M-r)(M+(1-r)N-2)!}{(N-1)!(M-rN)!} = \binom{M+(1-r)N-2}{N-1} + r\binom{M+(1-r)N-2}{N-2} \quad .$$

Although we were not yet able to find a closed form formula for the generating function $F(M,N,r,q)$, we found out (using Maple) some initial terms (according to N) of it. In Section 3 of this paper, we will present the first few terms corresponding to $N \leq 5$.

In the last section, we will present some possible future work to be done as well as connections with other combinatorial objects.

2.1. The recurrence relation for $F(M,N,r,q)$

It is not hard to come up with a recurrence relation for $F(M,N,r,q)$. Given an r-partition of n with the first part equal to M and exactly N parts, we can knock off the first row (we know by doing this we take away a factor of q^M) and what is left is an r-partition of $n-M$ with the first part equal to M_1 and exactly $N-1$ parts, where $1 \leq M1 \leq M-r$. Therefore we have the following recurrence relation:

$$F(M,N,r,q) = q^M \sum_{M_1=1}^{M-r} F(M_1, N-1, r, q)$$

It is clear that $F(M,1,r,q) = q^M$. With this information, we could program the procedure $\mathbf{F(M,N,r,q)}$ in Maple, which allows us to input specific M,N,r (q can be symbolic), and it will output the corresponding generating function.

2.2. Using Maple to discover (and prove!) patterns for $F(M,N,r,1)$

We start our experiment with $F(M,N,-1,1)$, which represents the total number of (-1)-partitions with the first part equal to M and exactly N parts. By fixing N and varying M, we generate the first 20 terms of $F(M,N,-1,1)$, which allows us to then use the procedure **GuessPol** in our package **rPar** to try guessing a polynomial for this sequence.

For example, typing $[\mathbf{seq(F(M1,1,-1,1),M1=1..20)}]$; yields

$$[1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1]$$

To guess a polynomial for this sequence, type: $\mathbf{GuessPol([seq(F(M1, 1, -1, 1), M1 = 1..20)], M, 1)}$;
Not surprisingly, it yields the constant polynomial 1.

Now try $N = 2$. Typing $[\mathbf{seq(F(M1, 2, -1, 1), M1 = 1..20)}]$; yields

$$[2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21]$$

$\mathbf{GuessPol([seq(F(M1, 2, -1, 1), M1 = 1..20)], M, 1)}$; yields

$$M + 1$$

Also not a surpise. Moving right along,

$\mathbf{GuessPol([seq(F(M1, 3, -1, 1), M1 = 1..20)], M, 1)}$; yields

$$\frac{(M + 4)(M + 1)}{2}$$

$\mathbf{GuessPol([seq(F(M1, 4, -1, 1), M1 = 1..20)], M, 1)}$; yields

$$\frac{(M + 6)(M + 5)(M + 1)}{6}$$

$\mathbf{GuessPol([seq(F(M1, 5, -1, 1), M1 = 1..20)], M, 1)}$; yields

$$\frac{(M + 8)(M + 7)(M + 6)(M + 1)}{24}$$

$\mathbf{GuessPol([seq(F(M1, 6, -1, 1), M1 = 1..20)], M, 1)}$; yields

$$\frac{(10 + M)(9 + M)(8 + M)(7 + M)(M + 1)}{120}$$

Without much effort, one can conjecture the following:

$$F(M, N, -1, 1) = \frac{(M + 1)(M + 2N - 2)!}{(N - 1)!(M + N)!} \quad .$$

Now let us experiment with $r = -2$.

$\mathbf{GuessPol([seq(F(M1, 1, -2, 1), M1 = 1..20)], M, 1)}$; yields

$$1$$

$\mathbf{GuessPol([seq(F(M1, 2, -2, 1), M1 = 1..20)], M, 1)}$; yields

$$M + 2$$

$\mathbf{GuessPol([seq(F(M1, 3, -2, 1), M1 = 1..20)], M, 1)}$; yields

$$\frac{(M + 7)(M + 2)}{2}$$

$\mathbf{GuessPol([seq(F(M1, 4, -2, 1), M1 = 1..20)], M, 1)}$; yields

$$\frac{(M+10)(M+9)(M+2)}{6}$$

$\mathbf{GuessPol([seq(F(M1, 5, -2, 1), M1 = 1..20)], M, 1)}$; yields

$$\frac{(M+13)(M+12)(M+11)(M+2)}{24}$$

$\mathbf{GuessPol([seq(F(M1, 6, -2, 1), M1 = 1..20)], M, 1)}$; yields

$$\frac{(M+16)(M+15)(M+14)(M+13)(M+2)}{120}$$

Again without much effort, one can conjecture that

$$F(M,N,-2,1) = \frac{(M+2)(M+3N-2)!}{(N-1)!(M+2N)!} \quad .$$

Comparing these two guesses, one can easily conjecture the formula for a general r:

$$F(M,N,r,1) = \frac{(M-r)(M+(1-r)N-2)!}{(N-1)!(M-rN)!} \quad .$$

Now, how do we prove this conjecture?

Recall that we programmed $\mathbf{F(M, N, r, q)}$ using the recurrence relation

$$F(M,N,r,q) = q^M \sum_{M_1=1}^{M-r} F(M_1, N-1, r, q)$$

and the initial condition $F(M,1,r,q) = q^M$. Note that $F(M,N,r,q)$ can be fully defind by this information. In other words, if we have found a formula that satisfies this recurrence relation and initial condition, then it *is* the formula for $F(M,N,r,q)$. This also applies to our current case when $q = 1$.

Therefore we went ahead and programmed the procedure $\mathbf{checkF(M, N, r)}$ and hooray! Maple was able to show that, by using symbolic computation, our conjectured formula for $F(M,N,r,1)$ indeed satisfies the recurrence relation (for $q = 1$). It is also easy to verify that the initial condition is satisfied. Therefore we have arrived, with a lot of help from Maple, at the following theorem:

Theorem 1.

$$F(M,N,r,1) = \frac{(M-r)(M+(1-r)N-2)!}{(N-1)!(M-rN)!} \quad .$$

Now the next step is to try to conjecture a formula for $F(M,N,r,q)$.

3.1. Can we find a pattern for $F(M, N, r, q)$?

Just like with the case $q = 1$, before we get so bold to go figuring out a pattern for $F(M, N, r, q)$, let us start by trying to figure out a pattern for $F(M, N, -1, q)$. Below are the guesses (using the **qGuessPol** procedure) for $N \leq 5$:

$\mathbf{qGuessPol([seq(F(M1, 1, -1, q), M1 = 1..20)], M, q, 1)}$; yields

$$q^M$$

$\mathbf{qGuessPol([seq(F(M1, 2, -1, q), M1 = 1..20)], M, q, 1)}$; yields

$$\frac{q^{M+1}(q^{M+1} - 1)}{(q - 1)}$$

$\mathbf{qGuessPol([seq(F(M1, 3, -1, q), M1 = 1..20)], M, q, 1)}$; yields

$$\frac{q^{M+2}(q^{M+3} + q^2 - q - 1)(q^{M+1} - 1)}{(q - 1)^2(q + 1)}$$

$\mathbf{qGuessPol([seq(F(M1, 4, -1, q), M1 = 1..20)], M, q, 1)}$; yields

$$\frac{q^{M+3}(q^{2M+8} + q^{M+7} - q^{M+5} - q^{M+4} - q^{M+3} + q^6 - 2q^4 - q^3 + 2q + 1)(q^{M+1} - 1)}{(q - 1)^3(q + 1)(q^2 + q + 1)}$$

$\mathbf{qGuessPol([seq(F(M1, 5, -1, q), M1 = 1..20)], M, q, 1)}$; yields

$$\frac{q^{M+4}(q^{M+5} + q^4 - q - 1)(q^{2M+10} - q^{M+4} - q^{M+3} + q^8 - q^5 - 2q^4 + 2q + 1)(q^{M+1} - 1)}{(q - 1)^4(q + 1)^2(q^2 + 1)(q^2 + q + 1))}$$

We can again **prove** that they are true by using the recurrence relation

$$F(M, N, r, q) = q^M \sum_{M_1=1}^{M-r} F(M_1, N - 1, r, q)$$

and the initial condition $F(M, 1, r, q) = q^M$, setting $r = -1$.

Note that, although the formulas above look like rational functions, they are in fact polynomials. It is easy to see why the first two formulas are polynomials. The rest are also polynomials simply because of the recurrence relation (a summation of polynomials is also a polynomial). So far we have not yet been able to find a closed form for $F(M, N, -1, q)$.

4.1. Future work and connection to other combinatorial objects

Interestingly, there is a direct connection between $F(M,N,-1,1)$ and Catalan's triangle (thanks again to **OEIS** to help us find such a connection). Catalan's triangle is a number triangle whose entries $C(n,k)$ is the number of strings consisting of n X's and k Y's such that no initial segment of the string has more Y's than X's. It satisfies the following:

$$C(n,k) = \binom{n+k}{k} - \binom{n+k}{k-1}$$

$$C(n,k) = \frac{(n+k)!(n-k+1)}{k!(n+1)!}$$

$$C(n+1,k) = C(n+1,k-1) + C(n,k)$$

Since we have shown $F(M,N,-1,1) = \frac{(M-r)(M+(1-r)N-2)!}{(N-1)!(M-rN)!}$, it is clear that $F(M,N,-1,1) = C(M+N-1,N-1)$. This turns out to be more or less obvious by a geometric interpretation. If one thinks in terms of lattice path, $C(M+N-1,N-1)$ is the number of lattice paths from the origin to the point $(M+N-1,N-1)$ that do not go above the line $y=x$ in the xy-plane and with $N-1$ steps up. (Note in each step we are only allowed to move right or up one step.) Each such path corresponds uniquely to a (-1)-partition with the first part exactly equal to M and exactly N parts. The part of the path that is above the line $y=N-2$ can take $(M+1)$ shapes. The number of horizontal steps in the shape determines the second part of the corresponding (-1)-partition (note there are $(M+1)$ possibilities for the second part). If there are k horizontal steps, then the second part of the partition is $M-k+1$. Similarly, the third part of the partition is determined by how many horizontal steps we take in the lattice path above line $y=N-3$ and below (including) the line $y=N-2$.

There also seems to be a bit of connection between the standard Young Tableau and $F(M,N,-1,1)$. For example, typing the sequence $[seq(F(5,N,-1,1),N=1..20]$ into **OEIS**, we will find it can also represent the number of standard Young Tableau of shape $(N+3,N-2)$. (A003517) If we change the value of M, then we can find correspondence with other standard Young Tableau. We conjecture that $F(M,N,-1,1)$ is equal to the number of standard Young Tableau of shape $(N+\lceil M/2\rceil, N-\lfloor M/2\rfloor)$.

For general r, we haven't found nice connections yet.