Relaxed Partitions

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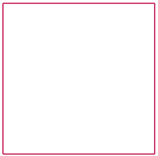

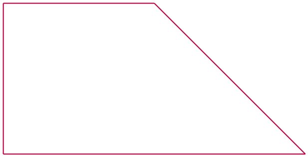

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Q: For a fixed r, how many r-partitions of integer n do we have? Noticing some reccurrence relations, we used Maple to program a procedure which we called $NPr(n, r)$ and it returns the number of r-partitions of n for any n and r. Below are the first 20 terms of $NPr(n, -1)$: number of (-1)-partitions of n:

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Now, can we find a generating function for a given r? The answer turned out to be yes! We typed the above sequence produced by $NPr(n, -1)$ into the OEIS, we found that its generating function seemed to be the reciprocal of

$$\sum_{k=0}^{\infty} \frac{(-1)^k q^{k^2}}{(q,q)_k}$$

.

Proof using bijection and generating functions

Since there is no proof for this formula provided in the OEIS and we could not find it elsewhere, we attempted to prove it by ourselves, and succeeded.

Theorem 1 [Zeilberger, 2018]

$$\sum_{k=0}^{\infty} \frac{(-1)^k q^{k^2}}{(q,q)_k} \sum_{n=0}^{\infty} NPr(n,-1)q^n = 1$$

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- The second sum is equal to $\sum_{L2} q^{sum(L2)}$ where L2 is a (-1)-partition.

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- Now take the weight of the whole set $S := \{(L1, L2)\}$ for all possible L1 and L2 by adding up the weight:

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- On one hand, the total weight should equal the lefthand of Theorem 1, that is:

$$\sum_{(L1,L2)\in S} (-1)^k q^{sum(L1)+sum(L2)} = \sum_{L1} (-1)^k q^{sum(L1)} \sum_{L2} q^{sum(L2)}$$

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- On the other hand, the total weight should equal to 1, which is the weight of a pair of empty partitions (,) (Note $(-1)^0 q^0 q^0 = 1$). Why?

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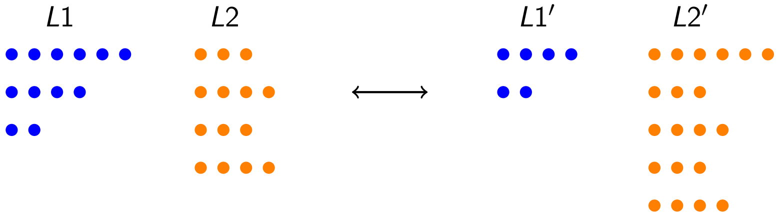
  This is because we can build a bijection between S={(L1,L2)} with itself such that if (L1,L2) is mapped to (L1',L2') under this bijection, then weight(L1,L2)=−weight(L1',L2'). Also, (L1,L2) is mapped to itself if and only if (L1,L2)=(,). In this case weight(L1,L2)=1. Therefore, we can pair up the elements in S so that their weight cancel each other except for the stand alone element (,). Thus weight(S)=1.

The bijection and generalization

The bijection: let a be the first part of the partition L1 and b be the first part of the partition L2. If $a \geq b - 1$ then move a from L1 to the first row of L2, otherwise, move b from L2 to the first row of L1. (If both L1 and L2 are empty then we do nothing.) For example, if the original pair (L1,L2) is $\{[6,4,2],[3,4,3,4]\}$, then it becomes $\{[4,2], [6,3,4,3,4]\}$. Clearly, under this bijection the number of parts of L1 either increase by 1 or decrease by 1, so weight(L1,L2)=−weight(L1',L2'). This finishes our proof.



Relaxed partitions with first part$=M$ and exactly N parts

Theorem 2 [Y, 2018] (Generalization of Theorem 1)

Given a negative integer r, the generating function for the number of r-partitions of n is the reciprocal of

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What if we restrict the first part to be M and the number of parts to be exactly N? Let us call this generating function F(M,N,r,q).

Reccurrence Relation

It is not hard to come up with a recurrence relation for $F(M, N, r, q)$:

$$F(M, N, r, q) = q^M \sum_{M_1=1}^{M-r} F(M_1, N-1, r, q)$$

Conjecture and proof using Maple

Knowing the reccurence relation, we programmed in Maple and found the following:

Typing $[\mathbf{seq}(\mathbf{F}(\mathbf{M1}, \mathbf{1}, -\mathbf{1}, \mathbf{1}), \mathbf{M1} = \mathbf{1}..\mathbf{20})]$; yields

$$[1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]$$

To guess a polynomial for this sequence, type:
$\mathbf{GuessPol}([\mathbf{seq}(\mathbf{F}(\mathbf{M1}, \mathbf{1}, -\mathbf{1}, \mathbf{1}), \mathbf{M1} = \mathbf{1}..\mathbf{20})], \mathbf{M}, \mathbf{1})$;
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Now try $N = 2$. Typing $[\mathbf{seq}(\mathbf{F}(\mathbf{M1}, \mathbf{2}, -\mathbf{1}, \mathbf{1}), \mathbf{M1} = \mathbf{1}..\mathbf{20})]$; yields

$$[2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21]$$

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$\mathbf{GuessPol([seq(F(M1, 6, -1, 1), M1 = 1..20)], M, 1)}$; yields

$$\frac{(10+M)(9+M)(8+M)(7+M)(M+1)}{120}$$

Conjecture and proof using Maple

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Comparing these two guesses, one can easily conjecture the formula for a general r:

$$F(M,N,r,1) = \frac{(M-r)(M+(1-r)N-2)!}{(N-1)!(M-rN)!} \quad .$$

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$$F(M, N, r, q) = q^M \sum_{M_1=1}^{M-r} F(M_1, N-1, r, q)$$

and the initial condition $F(M, 1, r, q) = q^M$. Note that $F(M, N, r, q)$ can be fully defind by this information. In other words, if we have found a formula that satisfies this recurrence relation and initial condition, then it *is* the formula for $F(M, N, r, q)$. This also applies to our current case when $q = 1$.

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Therefore we used Maple to verify both this equation and the intial condition are satisfied. So our conjectured formula F(M,N,r,1) was proved.

Conjecture and proof using Maple

Theorem 3 [Y, 2018]

$$F(M,N,r,1) = \frac{(M-r)(M+(1-r)N-2)!}{(N-1)!(M-rN)!}$$

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Now the next step is to try to conjecture a formula for $F(M,N,r,q)$.

Can we find a pattern for $F(M, N, r, q)$?

This turns out to be not so easy. Below are the guesses for $r = -1$ and $N <= 5$:
$\mathbf{qGuessPol}([\mathbf{seq}(\mathbf{F}(\mathbf{M1}, \mathbf{1}, -\mathbf{1}, \mathbf{q}), \mathbf{M1} = \mathbf{1..20})], \mathbf{M}, \mathbf{q}, \mathbf{1});$ yields

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$$\frac{q^{M+4}(q^{M+5} + q^4 - q - 1)(q^{2M+10} - q^{M+4} - q^{M+3} + q^8 - q^5 - 2q^4 + 2q + 1)(q^{M+1} - 1)}{(q-1)^4(q+1)^2(q^2+1)(q^2+q+1)}$$

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We can again **prove** that they are true by using the recurrence relation. Note that, although the formulas above look like rational functions, they are in fact polynomials. It would be very nice to find a general pattern for $F(M, N, r, q)$.

Some observations

Drew Sills's Observation: $F(M, N, -1, q)$ has denominator $(q; q)_N$ and a numerator of degree $N(M + N - 1)$. Thus it is plausible that the numerator is a (possibly alternating) sum of polynomials that are a power of q times a Gaussian polynomial of the form $G(M, N) := GP(2N + M - 1, N)$.

$$GP(m, r) := \frac{(q^{m-r+1}, q)_r}{(q, q)_r}$$

So far we have not made much progress in this, but we noticed an interesting pattern.

Some observations

$$\begin{cases} G(2,1) = q^2 + q + 1 \\ F(2,2,-1,q) = q^5 + q^4 + q^3 \end{cases}$$

$$\begin{cases} G(2,2) = \underline{q^6 + q^5 + 2q^4 + 2q^3 + 2q^2 + q} + 1 \\ F(2,3,-1,q) = \underline{q^9 + q^8 + 2q^7 + 2q^6 + 2q^5 + q^4} \end{cases}$$

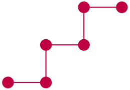
$$\begin{cases} G(2,3) = \underline{q^{12} + q^{11} + 2q^{10} + 3q^9 + 4q^8 + 4q^7 + 5q^6 + 4q^5} + 4q^4 \\ \qquad\qquad +3q^3 + 2q^2 + q + 1 \\ F(2,4,-1,q) = \underline{q^{14} + q^{13} + 2q^{12} + 3q^{11} + 4q^{10} + 4q^9 + 5q^8} \\ \qquad\qquad \underline{+4q^7} + 3q^6 + q^5 \end{cases}$$

Connection to other combinatorial objects

- Interestingly, there is a direct connection between $F(M, N, -1, 1)$ and Catalan's triangle (thanks again to **OEIS** to help us find such a connection). Catalan's triangle is a number triangle whose entries $C(n, k)$ is the number of strings consisting of n X's and k Y's such that no initial segment of the string has more Y's than X's. $F(M, N, -1, 1) = C(M + N - 1, N - 1)$.

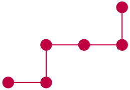
Connection to other combinatorial objects

Example: $M = 1, N = 3$



(1,1,1)

(1,2,2)



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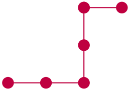

(1,1,2)

(1,2,3)

Connection to other combinatorial objects

- There also seems to be a bit of connection between the standard Young Tableau and $F(M, N, -1, 1)$. For example, typing the sequence $[seq(F(5, N, -1, 1), N = 1..20]$ into **OEIS**, we will found it can also represent the number of standard Young Tableau of shape $(N + 3, N - 2)$. (A003517)
- We conjecture that $F(M, N, -1, 1)$ is equal to the number of standard Young Tableau of shape $(N + \lceil M/2 \rceil, N - \lfloor M/2 \rfloor)$.

Thank you!