RESEARCH STATEMENT

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1. Introduction

While I am broadly interested in all kinds of mathematics, my graduate study and research focus is in combinatorics. In particular, I study patterns in permutations and words, and more recently, integer partitions. Side note: my favorite area of math in high school is geometry (I studied in China before college) and my undergraduate favorite area is abstract algebra. But at this moment, it is experimental math and combinatorics! I find it a lot of fun to discover patterns using computers and then try to prove them by various means (or leave them as conjectures). Below I will summarize the main projects that I have worked on during my graduate school years and possibilities for future work. **Many of them are accessible to undergraduate students and I will be happy to collaborate with students in my research.** The first three projects (2.1.-2.3.) are especially of current interest. (For a more detailed research statement please see my website: `http://sites.math.rutgers.edu/~my237/`.)

2. Projects

2.1. **Systematic counting of restricted partitions.** [This is joint work with Doron Zeilberger]

Project link: `http://sites.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/rpr.html`

One of the cornerstones of enumerative combinatorics (and number theory!) are integer partitions. A partition of a non-negative integer n is a list of integers $(\lambda_1, \dots, \lambda_k)$ such that $\lambda_1 \geq \cdots \geq \lambda_k \geq 1$ and $\lambda_1 + \cdots + \lambda_k = n$.

First, we give a simple definition of a "partition-pattern", or "pattern" for short.

Definition 2.1. *A partition-pattern (pattern for short) is a list $a = [a_1, \dots, a_r]$ of length $r \geq 1$ of non-negative integers. A partition $\lambda = (\lambda_1, \dots, \lambda_k)$* **contains** *the pattern $a = [a_1, \dots, a_r]$ if there exists $1 \leq i \leq k - r$ such that*

$$\lambda_i - \lambda_{i+1} = a_1 \quad , \quad \lambda_{i+1} - \lambda_{i+2} = a_2 \quad , \quad \dots \lambda_{i+r-1} - \lambda_{i+r} = a_r \quad .$$

A partition **avoids** the pattern if it does not contain the pattern. A partition avoids the set of patterns A, if it **avoids** every pattern in A. For example, partitions whose adjacent parts differ by at least 2 is equivalent to partitions that avoid $\{[0], [1]\}$.

Goal: devise an efficient algorithm, that inputs an **arbitrary** set of patterns, P, and an arbitrary positive integer N, and outputs the first N terms of the sequence enumerating partitions of n avoiding the set of patterns P.

We used two approaches to attack this problem. One approach is to adapt the celebrated Goulden-Jackson ([2], [7]) method to this new context. The Goulden-Jackson method traditionally only deals with words. I was able to extend this method to partitions. This approach is of considerable *theoretical* interest, but it turned out to be less efficient than a more straightforward (which we call "positive") approach, where the basic idea is to cut off the largest part in the partition and get a system of recurrences and use dynamical programming. This algorithm can be made quadratic in time and memory. However, we were not content with just stopping here, so we went on to generalize this algorithm in order to deal with partitions with more specific restrictions, which also allowed us to efficiently search for new partition identities. This leads to the birth of the second project, which is described in Section 2.2.

Future work:
1. Make the algorithm in the "positive" approach quadratic in time and memory. This is achievable through substitution, subtraction and some other algebraic manipulations of recurrence relations. It shouldn't be too hard and is **accessible to undergraduate students.**

2. Simplify the algorithm in the "negative" approach.

2.2. **Searching for partition identities.** [Joint work with Matthew Russell]
Talk slides: `https://sites.math.rutgers.edu/~zeilberg/expmath/MingjiaY19.pdf`

Even though the previous project enables us to enumerate partitions avoiding an *arbitrary* (finite) set of patterns, many partition theoretic sum-sides of partition identities require more specific restrictions. One example (a Russell-Kanade conjecture, see [5]) is the following:

(a) No consecutive parts allowed.
(b) Odd parts do not repeat.
(c) Even parts appear at most twice.
(d) If a part $2j$ appears twice then $2j \pm 3, 2j \pm 2$ are forbidden to appear at all.
(e) 2+2 is not allowed as a sub-partition.

In order to generalize the efficient "positive" approach described at the end of 2.1., we introduce some new notions and formulate these partitions in a different way, which turns out to be very flexible and also easy to feed to a computer. Here are the three major parameters that we use:

A: the set of patterns to avoid "globally"
Mod: the list of patterns to avoid according to congruence conditions of the largest part of a sub-partition (for details please see `http://sites.math.rutgers.edu/%7Emy237/Pos_Ext`)
IC: the list of sub-partitions to avoid (we call this "initial conditions")

Let us look at the Kanade-Russell conjecture in light of this notation: no consecutive parts allowed translates to the global condition: $A = \{[1]\}$. Odd parts do not repeat, even parts appear at most twice, along with part (d) translate to $Mod = [\{[0,0],[0,3],[0,2],[2,0]\},\{[0],[3,0]\}]$. Part (e), that is, 2+2 is not allowed as a sub-partition translates to IC=[[2,2]].

Using these new notations I came up with an algorithm that enables very efficient computation. Let $GP(m,n,A,Mod,B,IC)$ be the number of partitions of n, with largest part m, and the restrictions A, Mod, IC described above, and B and B'' are beginning restrictions. At the core of the algorithm is the recurrence relation:

$$GP(m,n,A,Mod,B,IC) = \sum_{\substack{1\le m'\le m\\ [m-m']\notin A\cup B'\cup Mod[i+1]}} GP(m',n-m,A,Mod,B'',IC) \quad .$$

With this algorithm and making use of Frank Garvan's q-series Maple package, we are currently conducting the search over various parameter restrictions, with the help of **Amarel cluster computing** `https://oarc.rutgers.edu/amarel/`. We have already found many seemingly new Rogers-Ramanujan type identities, and has generalized one of them to an infinite family.

Future work (all of them are appropriate for collaboration with undergraduate or master's students) :
1. Put more variations on the parameters to enlarge our search space.

2. Currently our approach only deals with conditions on contiguous sub-partitions. It will be nice to develop a general frame work/an efficient way to search for identities that avoid sub-partitions that are not necessarily contiguous.

3. During my talk at 2019 AMS Fall Southeastern Sectional Meeting, Drew Sills (see `http://home.dimacs.rutgers.edu/~asills/` and Ali Uncu (see `https://risc.jku.at/m/ali-uncu/`) offered some great suggestions to make our algorithm more flexible, including allowing the initial conditions to contradict the global conditions and allowing Nandi-type conditions. We are currently working on implementing these.

2.3. Relaxed Partitions.

Project link: http://sites.math.rutgers.edu/~my237/RP

Talk slides: http://sites.math.rutgers.edu/~my237/Auburn.pdf

This is a project that can be extended in many directions. I believe this project and its various related projects could be a lot of fun for undergraduate students.

In this project, we took a road less traveled and studied an object which we call "relaxed partitions", or more specifically, r-partitions with r to be specified. Unlike the traditional partitions where we require $\lambda_i - \lambda_{i+1} \geq 0$, for r-partitions we require $\lambda_i - \lambda_{i+1} \geq r$ where r can be negative. For example, $(2, 3, 1, 1)$ is a (-1)-partition of 7.

Just as with traditional partitions, there are many questions one could ask about r-partitions. Perhaps one of the first questions to ask is: "for a fixed r, how many r-partitions of the integer n do we have?" Using an easy recurrence relation, I programmed $\mathbf{NPr(n, r)}$, which answers this question for specific n and r. But can we find a generating function for a given r? The answer turned out to be yes! And there is a nice closed form for it. By typing in a sequence of entries produced by $\mathbf{NPr(n, -1)}$ into the **OEIS**, we found that it is the reciprocal of

$$\frac{\sum_{k=0}^{\infty} (-1)^k q^{k^2}}{\prod_{i=1}^{k} (1 - q^i)} \quad .$$

Recognizing this is the generating function for the "weighted" (with weight $(-1)^k$) number of partitions (traditional) of integer n into parts with difference at least 2, Doron Zeilberger (2018) provided a short and elegant bijective proof for it. I generalized this to general r:

Theorem 2.2. *(Yang, 2018) The generating function for the number of r-partitions of n is the reciprocal of*

$$\frac{\sum_{k=0}^{\infty} (-1)^k q^{k(2+(1-r)(k-1))/2}}{\prod_{i=1}^{k} (1 - q^i)} \quad ,$$

which is the "weighted" number of partitions of integer n into parts with difference at least $(-r+1)$.

Since this question was answered, we turned our focus to restricted r-partitions such that the first part and the number of the parts are fixed. Let $a_r(M, N, n)$ be the number of r partitions with the first part equal to M and exactly N parts. To go with the notation in our Maple package **rPar**, let $Ftr(M, N, r, q)$ be the generating function for $a_r(M, N, n)$. Using a simple recurrence relation $Ftr(M, N, r, q)$ satisfies, I was able to program it in Maple and happily used Maple to conjecture (**and prove!**) the closed form for the case $q = 1$ (i.e., the total number of r partitions with the first part equal to M and exactly N parts). It is:

Theorem 2.3. *(Yang, 2018)*

$$Ftr(M,N,r,1) = \frac{(M-r)(M+(1-r)N-2)!}{(N-1)!(M-rN)}$$

$$= \binom{M+(1-r)N-2}{N-1} + r\binom{M+(1-r)N-2}{N-2}$$

Although I was not yet able to find a closed form formula for the generating function $Ftr(M,N,r,q)$, I found out (using Maple) some initial terms (according to N) of it. Below are the first four terms produced by our Maple program, which can be proven easily using a recurrence relation. Note even though they look like rational functions, they are in fact polynomials.

$\mathbf{qGuessPol([seq(F(M1,1,-1,q),M1=1..20)],M,q,1)}$; yields

$$q^M$$

$\mathbf{qGuessPol([seq(F(M1,2,-1,q),M1=1..20)],M,q,1)}$; yields

$$\frac{q^{M+1}(q^{M+1}-1)}{(q-1)}$$

$\mathbf{qGuessPol([seq(F(M1,3,-1,q),M1=1..20)],M,q,1)}$; yields

$$\frac{q^{M+2}(q^{M+3}+q^2-q-1)(q^{M+1}-1)}{(q-1)^2(q+1)}$$

$\mathbf{qGuessPol([seq(F(M1,4,-1,q),M1=1..20)],M,q,1)}$; yields

$$\frac{q^{M+3}(q^{2M+8}+q^{M+7}-q^{M+5}-q^{M+4}-q^{M+3}+q^6-2q^4-q^3+2q+1)(q^{M+1}-1)}{(q-1)^3(q+1)(q^2+q+1)}$$

Drew Sills, during his short visit to Rutgers and our brief meeting, observed that $F(M,N,-1,q)$ has denominator $(q;q)_N$ and a numerator of degree $N(M+N-1)$. He pointed out that it is plausible the numerator is a (possibly alternating) sum of polynomials that are a power of q times a Gaussian polynomial of the form $G(M,N) := GP(2N+M-1,N)$, where

$$GP(m,r) := \frac{(q^{m-r+1},q)_r}{(q,q)_r}.$$

So far we have not made much progress in this, but this led me to discover an interesting pattern:

$$\begin{cases} G(2,1) = q^2+q+1 \\ F(2,2,-1,q) = q^5+q^4+q^3 \end{cases}$$

$$\begin{cases} G(2,2) = \underline{q^6+q^5+2q^4+2q^3+2q^2+q}+1 \\ F(2,3,-1,q) = \underline{q^9+q^8+2q^7+2q^6+2q^5+q^4} \end{cases}$$

$$\begin{cases} G(2,3) = \underline{q^{12}+q^{11}+2q^{10}+3q^9+4q^8+4q^7+5q^6+4q^5}+4q^4 \\ \qquad\qquad +3q^3+2q^2+q+1 \\ F(2,4,-1,q) = \underline{q^{14}+q^{13}+2q^{12}+3q^{11}+4q^{10}+4q^9+5q^8} \\ \qquad\qquad \underline{+4q^7}+3q^6+q^5 \end{cases}$$

Notice for each pair, the underlined parts have the same coefficients. This pattern continues indefinitely for bigger M and N. This is an ongoing project to explore why this intriguing pattern exists and if it can point us to discover a closed form for $F(M,N,-1,q)$, or even $F(M,N,r,q)$.

Connection to other combinatorial objects:
With the help of **OEIS**, I found a direct connection between $F(M,N,-1,1)$ and Catalan's triangle. Since $F(M,N,-1,1) = \frac{(M-r)(M+(1-r)N-2)!}{(N-1)!(M-rN)!}$, it is clear that $F(M,N,-1,1) = C(M+N-1,N-1)$. And there turned out to be a nice a geometric interpretation.

There also seems to be a bit of connection between the standard Young Tableau and $F(M,N,-1,1)$. For example, typing the sequence $[seq(F(5,N,-1,1),N=1..20]$ into **OEIS**, we will find it can also represent the number of standard Young Tableau of shape $(N+3,N-2)$. (A003517) If we change the value of M, then we can find correspondence with other standard Young Tableau. I conjecture that $F(M,N,-1,1)$ is equal to the number of standard Young Tableau of shape $(N+\lceil M/2\rceil, N-\lfloor M/2\rfloor)$. For general r, we haven't found nice connections yet.

Future work: As mentioned earlier, it would be wonderful if we could discover why the intriguing pattern between the F and G exists. This may enable us to find a closed form for $F(M,N,-1,q)$, and perhaps give us insights for $F(M,N,r,q)$. During my talk at 2019 AMS Spring Southeastern Sectional Meeting, Tim Huber (see `https://faculty.utrgv.edu/timothy.huber/`) mentioned that these generating functions' approximation of the Gaussian polynomials may indicate they count permutations by some statistic and may arise from a quotient of generating functions of a special form. This is still under exploration. It would also be nice if we can find connection between $F(M,N,r,1)$ and other combinatorial objects. I am also interested in exploring plane partitions and "fractional counting", some initial Maple "playing-around" can be found on `http://sites.math.rutgers.edu/~my237/RP`.

Below are two projects related to permutations and words, which are also interesting, but less recent. I only list them here. For details, please see the long version of my research statement on my website: `http://sites.math.rutgers.edu/~my237/`.

2.4. **Increasing Consecutive Patterns in Words.** (2018)
Project link: `http://sites.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/icpw.html`
J. Algebr. Comb. (2019). `https://link.springer.com/article/10.1007/s10801-018-0868-5`

2.5. **Enumeration of words that contain pattern 123 exactly once.** (2017)
Project link: `http://sites.math.rutgers.edu/~my237/One123`
Ann. Comb. 23 (2019) 207–217. `https://link.springer.com/article/10.1007/s00026-019-00416-z`

References

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[2] Ian Goulden and David Jackson, An inversion theorem for cluster decomposition of sequences with distinguished subsequences, *J. London Math. Soc.(2)* **20** (1979), 567-576.

[3] S. Kanade and M. C. Russell, IdentityFinder and some new identities of Rogers-Ramanujan type, *Exp. Math.* **24** (2015), 419-423. `https://arxiv.org/abs/1411.5346`.

[4] S. Kanade, D. Nandi and M. C. Russell, A variant of IdentityFinder and some new identities of Rogers-Ramanujan- MacMahontype. `https://arxiv.org/abs/1902.00790`.

[5] S. Kanade and M. C. Russell, Staircases to analytic sum-sides for many new integer partition identities of Rogers-Ramanujan type. *Electron. J. Combin.*, **26(1)**:Paper 1.6, 2019. `https://arxiv.org/pdf/1803.02515`.

[6] B. Nakamura, Computational approaches to consecutive pattern avoidance in permutations, *Pure Math. Appl.* (PU.M.A.) **22** (2011), 253–268.

[7] J. Noonan and D. Zeilberger, The Goulden-Jackson cluster method: extensions, applications, and implementations, *J. Difference Eq. Appl.* **5** (1999), 355–377.
`http://sites.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/gj.html` .

[8] N. J. A. Sloane, *The On-Line Encyclopedia of Integer Sequences*, `http://oeis.org`.

[9] Andrew V. Sills, *An Invitation to the Rogers-Ramanujan Identities*, CRC Press, Boca Raton, 2018.

(Here I omitted the references related to project 2.4 and 2.5. For a complete list, please see the long version of the research statement on my website.)