A Modern Approach to Gårding's Asymptotics Result

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April 27, 2018

Abstract

In this note, we discuss a modern approach to Lars Gårding's 1953 result on the asymptotics of eigenvalues of elliptic operators. When the paper was originally written, a lot of the estimates and detailed constructions needed to be done by hand. However, using the modern theory of pseudodifferential operators, most of these can be abstracted to the general theory, allowing the results to be presented more concisely and put in a more general context. [1]

1 Introduction

The main result that we seek to reprove in this note is presented in [2]. Let $a(x, D)$ be a linear partial differential operator of order $2m$

$$a(x, D)u(x) = \sum_{|\alpha| \leq 2m} a_\alpha(x) D^\alpha u(x)$$

where $D^\alpha = i^{|\alpha|} \frac{\partial^{|\alpha|}}{\partial x^\alpha}$. We assume that the coefficients a_α are smooth in a region $T \subset \mathbb{R}^n$ which contains our domain of interest Ω, and that a is elliptic in the sense that

$$a_0(x, \xi) = \sum_{|\alpha| = 2m} a_\alpha(x) \xi^\alpha$$

is a positive definite polynomial in ξ for all $x \in T$. We say that λ is a Dirichlet eigenvalue of a for the domain Ω if there exists a function $u \in H_0^m(\Omega)$ so that $a(x, D)u = \lambda u$. The final result in this paper is that $N(\lambda)$, the number of Dirichlet eigenvalues of a less than λ, satisfies

$$N(\lambda) \sim (2\pi)^{-n} \lambda^{n/2m} \iint_{a_0(x,\xi)<1} d\xi \ dx$$

As an initial application, consider the operator $a = -\Delta$, the standard Laplacian on the domain Ω. For this operator, the polynomial a_0 is $a_0(x, \xi) = |\xi|^2$, so that $a_0(x, \xi) < 1$ if and only if ξ is in the unit ball in $\mathbb{R}^n$. Then, we have that

$$\iint_{a_0(x,\xi)<1} d\xi \ dx = \int_\Omega dx \int_{B_1(0)} d\xi = \text{vol}(\Omega)\omega_n$$

Since $m = 1$ in this case, Gårding's result reduces to

$$N(\lambda) \sim (2\pi)^{-n} \lambda^{n/2} \omega_n \text{vol}(\Omega)$$

which is exactly the Weyl asymptotics of the Dirichlet eigenvalues of the Laplacian. The Gårding result thus extends the Weyl asymptotic result to all elliptic operators.

[1] This was generated for a class in Several Complex Variables, so there may be some flaws in these arguments. I know very little about pseudodifferential operators and only somewhat understand what parametrices do.

This note will seek to be a self-contained discussion of this result. Therefore, Section 2 will discuss the background material necessary to read the rest of the note. Section 3 will talk about elliptic operators in general and discuss Gårding's previous result that will allow us to discuss the Green's Function operator, which serves as the inverse to a differential operator. Section 4 will introduce the basic definitions from the theory of pseudodifferential operators, focusing on the particular definitions needed for this problem. Finally, Section 5 will use all of this information to attack Gårding's asymptotic estimate.

2 Background Information

2.1 Fourier Transforms

Definition 2.1. The *Fourier Transform* of a function f is given by

$$\hat{f}(\xi) = (2\pi)^{-n} \int e^{ix\cdot\xi} f(x)\ dx$$

The inverse formula is given by

$$\check{g}(x) = \int e^{-ix\cdot\xi} g(\xi)\ d\xi$$

The following are standard properties of Fourier Transforms. See [1] for the proofs. Note that some of the notation here may differ from [1], but the results and proofs are still the same.

Proposition 2.1. *(a) There is a constant C_n depending only on the dimension so that*

$$||f||_2 = C_n ||\hat{f}||_2$$

(b) For any multi-index α, we have that $\widehat{D^\alpha f} = \xi^\alpha \hat{f}$ and $\widehat{x^\alpha f} = D^\alpha \hat{f}$ where $D^\alpha = i^{|\alpha|}\frac{\partial^{|\alpha|}}{\partial x^\alpha}$.

(c)

$$\int f(x)\overline{g(x)}\ dx = C_n \int \hat{f}(\xi)\overline{\hat{g}(\xi)}\ d\xi$$

(d) If $\hat{f} \in L^1(\mathbb{R}^n)$, then f is bounded and continuous.

Proof. See [1] for the proofs of parts (a) through (c).

(d) If $\hat{f}$ is L^1, then we can bound

$$|f(x)| \leq \int |e^{-ix\cdot\xi}\hat{f}(\xi)| d\xi \leq \int |\hat{f}(\xi)| d\xi = ||\hat{f}||_{L^1}$$

so that f is bounded. Furthermore, since $|f(y)| \leq ||\hat{f}||_{L^1}$, the Dominated Convergence theorem lets us pass the limit as $y \to x$ through the integral to get that f is continuous.

■

2.2 Sobolev Spaces

Definition 2.2. The Sobolev $H^m(\Omega)$ norm of a function $f \in C_0^\infty(\Omega)$ is defined as

$$||f||^2_{H^m(\Omega)} := \int_\Omega \sum_{|\alpha|\leq m} |D^\alpha f|^2\ dx$$

and the corresponding inner product is

$$\langle f, g\rangle_{H^m(\Omega)} := \int_\Omega \sum_{|\alpha|\leq m} D^\alpha f \overline{D^\alpha g}\ dx$$

Definition 2.3 (Classical Sobolev Spaces)**.** The Sobolev space $H^m(\Omega)$ is defined to be the set

$$H^m(\Omega) := \{u \in L^2(\Omega) \ | \ D^\alpha u \in L^2(\Omega) \text{ exists } \ \forall\, \alpha \leq m\}$$

The space $H_0^m(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in the $H^m(\Omega)$ norm.

Proposition 2.2. *The above definition of $H_0^m(\Omega)$ is equivalent to*

$$H_0^m(\Omega) = \{u \in H^m(\Omega) \ | \ D^\alpha u = 0 \ \text{ on } \partial\Omega \ \ \forall\, |\alpha| < m\}$$

Proof. We can define a trace operator $T : H^1(\Omega) \to L^2(\partial\Omega)$ that is continuous and, for $u \in C^1(\bar{\Omega})$, $Tu = u\mid_{\partial\Omega}$. Then, for u in the closure of $C_0^\infty(\Omega)$, there is a sequence $u_n \to u$ where the $u_n \in C_0^\infty(\Omega)$ and the convergence is in the L^2 norm on each derivative. In particular, this implies that $D^\alpha u_m \to D^\alpha u$ in $H^1(\Omega)$ for all $|\alpha| < m$. Thus

$$T(D^\alpha u) = \lim_{n\to\infty} T(D^\alpha u_n) = 0$$

For the other direction, we need to use the fact that $T(D^\alpha u) = 0$ to construct a sequence of smooth functions that converge to u. The proof can be found in [1]. ■

Proposition 2.3. *For any $f \in H^m(\mathbb{R}^n)$, the norms $||f||_{H^m(\mathbb{R}^n)}$ and $||(1 + |\xi|^2)^{m/2}\hat{f}||_{L^2(\mathbb{R}^n)}$ are equivalent.*

Proof. Taking the square of both sides, we can see that

$$||f||^2_{H^m(\mathbb{R}^n)} = \sum_{|\alpha|\leq m} ||D^\alpha f||^2_{L^2(\mathbb{R}^n)} = \sum_{|\alpha|\leq m} ||\xi^\alpha \hat{f}||^2_{L^2(\mathbb{R}^n)}$$

and this last sum can be expressed as

$$||f||_{H^m(\mathbb{R}^n)} = \int \sum_{|\alpha|\leq m} |\xi|^{2\alpha} |\hat{f}|^2 \ d\xi$$

while the second norm is

$$\int (1 + |\xi|^2)^m |\hat{f}(\xi)|^2 \ d\xi$$

Using the fact that

$$\frac{1}{m!}(1 + |\xi|^2)^m \leq \sum_{|\alpha|\leq m} |\xi^{2\alpha}| \leq m!(1 + |\xi|^2)^m$$

we easily get equivalence of these two norms. ■

As long as $\partial\Omega$ is smooth enough, then any function in $H^m(\Omega)$ can be extended to one in $H^m(\mathbb{R}^n)$ with a norm bound. Therefore, we can use Fourier Transform methods on functions in $H^m(\Omega)$ by first extending them to all of $\mathbb{R}^n$, using the Fourier Transform, and then restricting back to Ω. The next result uses that technique.

Proposition 2.4 (Sobolev Interpolation)**.** *Let $u \in H^m(\Omega)$. Then, for any $0 < k < m$ and any $\epsilon > 0$, there exists a constant $C_\epsilon > 0$ so that*

$$||u||_{H^k(\Omega)} \leq \epsilon ||u||_{H^m(\Omega)} + C_\epsilon ||u||_{L^2(\Omega)}$$

Proof. Consider the function, for any $\epsilon > 0$

$$(1 + |\xi|^2)^k - \epsilon(1 + |\xi|^2)^m$$

This function is continuous, and since $m > k$, goes to $-\infty$ as $|\xi| \to \infty$. Thus, it is bounded from above, and so there exists a constant C_ϵ so that

$$(1 + |\xi|^2)^k \leq \epsilon(1 + |\xi|^2)^m + C_\epsilon$$

Then, for any function $u \in H^m(\mathbb{R}^n)$, we have that

$$\begin{aligned} ||u||^2_{H^k(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} (1 + |\xi|^2)^k |\hat{u}(\xi)|^2 \, d\xi \leq \int_{\mathbb{R}^n} [\epsilon(1 + |\xi|^2)^m + C_\epsilon] |\hat{u}(\xi)|^2 \, d\xi \\ &= \int_{\mathbb{R}^n} \epsilon(1 + |\xi|^2)^m |\hat{u}(\xi)|^2 \, d\xi + \int C_\epsilon |\hat{u}|^2 \, d\xi = \epsilon ||u||^2_{H^m(\mathbb{R}^n)} + C_\epsilon ||u||_{L^2(\mathbb{R}^n)} \end{aligned}$$ ■

The same result holds on $H^m(\Omega)$ by extension.

Proposition 2.5 (Poincaré's Inequality)**.** *Let $u \in C^1_0(\Omega)$, with Ω bounded. Then there exists a constant $C(\Omega, n, p)$ so that*

$$||u||_{L^p(\Omega)} \leq C ||Du||_{L^p(\Omega)}$$

for any $1 \leq p < \infty$.

Proof. Since Ω is bounded, there exists a cube $[-R, R]^n$ so that $\Omega \subset [-R, R]^n$. Let $x = (x_1, x')$ be any point in Ω. Then since u has compact support in Ω,

$$|u(x)|^p = \left| \int_{-R}^{x} \frac{\partial u}{\partial x_1}(z, x') \, dz \right|^p \leq \left| \int_{-R}^{R} \frac{\partial u}{\partial x_1}(z, x') \, dz \right|^p \leq (2R)^{p/q} \int_{-R}^{R} \left| \frac{\partial u}{\partial x_1} \right|^p \leq (2R)^{p/q} \int_{-R}^{R} |Du(z, x')|^p$$

Then we integrate over the entire domain, and pick up another factor of $2R$ from the x_1 integral. ■

Corollary 2.1. *If $u \in H^m_0(\Omega)$, then there exists a constant $C(\Omega, m)$ so that*

$$||u||_{H^m(\Omega)} \leq C \sum_{|\alpha| = m} ||D^\alpha u||_{L^2(\Omega)}$$

Proof. Take $u \in C^\infty_0(\Omega)$. Since all derivatives of u are at least $C^1_0(\Omega)$, Poincaré's inequality applies, so that

$$||D^\alpha u||_{L^p} \leq C ||D(D^\alpha u)||_{L^p}$$

for any α. Applying this inductively gives that we can control any number of derivatives less than m by the mth order derivatives. The result follows by density of $C^\infty_0(\Omega) \subset H^m_0(\Omega)$. ■

3 Elliptic Operators

The most general linear partial differential operator of even order $2m$ has the form

$$a(x, D_x)u := \sum_{|\alpha| \leq 2m} a_\alpha(x) D^\alpha u$$

where the α are multi-indices

$$\alpha = (\alpha_1, ..., \alpha_n) \qquad D^\alpha = i^{|\alpha|} \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}$$

We say that such an operator is *elliptic* if the highest order terms satisfy

$$a_0(x, \xi) := \sum_{|\alpha| = 2m} a_\alpha(x) \xi^\alpha \neq 0$$

for all $\xi \neq 0 \in \mathbb{R}^n$ and all $x \in \Omega$. We say it is strictly elliptic if

$$a_0(x, \xi) \geq c ||\xi||^{2m}$$

for all $x \in \Omega$ and $\xi \in \mathbb{R}^n$.

3.1 Gårding's Inequality

Gårding's inequality is a result for elliptic partial differential operators. It gives some very nice results about the operator, as long as we can add a constant to it.

Definition 3.1. Let a be an elliptic partial differential operator of order $2m$. The bilinear form $B_a[u,v]$ on $H_0^m(\Omega)$ associated to a is an expression of the form

$$B_a[u,v] = \int \sum_{|\alpha|,|\beta|\leq m} a'_{\alpha,\beta}(x) D^\alpha u \overline{D^\beta v}\, dx$$

where the coefficient functions $a'_{\alpha,\beta}$ are chosen so that

$$B_a[u,v] = \int \left(\sum_{|\alpha|\leq 2m} a_\alpha(x) D^\alpha u \right) \overline{v}\, dx$$

for all $u, v \in C_0^\infty(\Omega)$ after integration by parts.

Theorem 3.1 (Gårding's Inequality)**.** *Let a be an elliptic partial differential operator of order $2m$. Let $B_a[u,v]$ be the bilinear form on $H_0^m(\Omega)$ associated with this operator. Then there exists a constant t_0 such that for all $t > t_0$, there exists a constant $c > 0$ so that*

$$B[u,u] + t||u||^2_{L^2(\Omega)} \geq c||u||^2_{H^m(\Omega)}$$

Proof. We can write $B_a[u,u]$ as

$$B_a[u,u] = \int_\Omega \left(\sum_{|\alpha|=2m} a_\alpha(x) \sum_{\substack{\beta+\gamma=\alpha \\ |\beta|=|\gamma|=m}} D^\beta u \overline{D^\gamma u} \right) + R(u)\, dx$$

where every term in $R(u)$ has, when adding up the number of derivatives on both factors of u, a total number of derivatives less than $2m$. Thus, each term in $R(u)$ has either two factors with strictly less than m, or one factor with m derivatives and one with strictly less. For terms with lower order derivatives, Sobolev Interpolation allows us to control

$$||u||^2_{H^k(\Omega)} \leq \epsilon ||u||^2_{H^m(\Omega)} + C_\epsilon ||u||_{L^2(\Omega)}$$

for any $\epsilon > 0$. This, combined with Cauchy Schwartz with ϵ allows us to take any term that does not have both derivatives of order m, and control it by a small multiple of $||u||_{H^m(\Omega)}$ plus a large multiple of $||u||_{L^2(\Omega)}$ The highest order term with both derivatives of order m can be bounded from below by $c_1||u||^2_{H^m(\Omega)}$ because the operator is elliptic, into which we can absorb all of the small $||u||_{H^m(\Omega)}$ terms to get the bound that we want. ■

3.2 Green's Operators

This result implies that for t large enough, the operator $a + t$, adding a tu term to the end of the operator, is a coercive operator. The Lax-Milgram theorem (which also post-dates this paper) guarantees solutions in the sense that for any linear functional ℓ on H_0^m, there exists a unique $u^\ell \in H_0^m$ so that

$$a_t(u^\ell, v) = \ell(v) \qquad \forall\, v \in H_0^m \qquad a_t(u,v) = B[u,v] + t\langle u, v\rangle_{L^2}$$

Defining

$$\ell(v) = \int v\bar{f}$$

and letting u^f denote the corresponding f, we have that

$$a_t(u^f, v) = \langle f, v \rangle$$

The map $f \to u^f$, denoted by G_t acts as an inverse to the differential operator a_t, in that

$$\langle f, g \rangle_{L^2} = a_t(G_t f, g) \qquad a_t G_t f = G_t a_t f = f$$

Our goal will be to show that this operator has an integral kernel which has nice properties. The main property we care about is that if λ is an eigenvalue of a, then $(\lambda + t)^{-1}$ will be the corresponding eigenvalue of G_t. This also works the other way around, that is, having information on the eigenvalues of G_t should tell us things about the eigenvalues of a.

3.3 Tauberian Theorem

In order to connect information about the series of $(\lambda+t)^{-1}$ for λ an eigenvalue of a to the distribution of the eigenvalues, we need to use a Tauberian theorem. For this we reference the work of Hardy and Littlewood [3]:

Theorem 3.2 (Theorem 4 [3]). *Suppose that* $f \geq 0$ *and* $f \in L^1_{loc}(0, \infty)$ *and that*

$$\frac{f(x)}{(x+t)^\rho} \in L^1(0, \infty)$$

for some (all) $t > 0$. *Suppose further that*

$$h(t) = \int_0^\infty \frac{f(x)\ dx}{(x+t)^\rho} \sim \frac{H}{t^\sigma}$$

where $0 < \sigma < \rho$, $H > 0$ *as* $t \to \infty$. *Then*

$$F(x) = \int_0^x f(u)\ du \sim \frac{H\Gamma(\rho)}{\Gamma(\sigma)\Gamma(\rho - \sigma + 1)} x^{\rho - \sigma}$$

when $x \to \infty$.

However, further work on these types of theorems from Pleijel [4] suggests that these results work on any measure space, not just standard $L^1(dx)$. In order to get something that fits with the results we will generate later for the eigenvalues, we apply the theorem above with the function $f \equiv 1$, where the measure space is taken to be the spectral counting measure, namely

$$\mu([0, \lambda]) = N(\lambda)$$

Theorem 3.3 (Modified Tauberian Theorem). *If*

$$\sum_k (\lambda_k + t)^{-\rho} < \infty$$

and

$$\sum_k (\lambda_k + t)^{-\rho} \sim \frac{H}{t^\sigma}$$

then

$$N(x) = \sum_{\lambda < x} 1 = \int_0^t f(u)\ d\mu(u) \sim \frac{H\Gamma(\rho)}{\Gamma(\sigma)\Gamma(\rho - \sigma + 1)} x^{\rho - \sigma}$$

This theorem gives the explicit connection between the eigenvalues of G_t^k and the asymptotics of the eigenvalues of a.

4 Pseudodifferential Operators

The last major piece of the puzzle for analyzing Gårding's asymptotic result in the modern setting is the theory of pseudodifferential operators. The introduction here is adapted from the corresponding chapters of [5].

4.1 Symbol Classes

To start consider a differential operator of the form

$$p(x,D)=\sum_{|\alpha|\le k}a_\alpha(x)D^\alpha$$

By properties of the Fourier Transform, we know that

$$p(x,D)f(x)=(2\pi)^{-n}\int p(x,\xi)\hat{f}(\xi)e^{ix\cdot\xi}\,d\xi \tag{1}$$

where

$$p(x,\xi)=\sum_{|\alpha|\le k}a_\alpha(x)\xi^\alpha$$

Therefore, we have that differential operators can be represented by smooth functions $p(x,\xi)$ in the form given above, where $p(x,\xi)$ is a polynomial in ξ in this case. Pseudodifferential operators are defined in the same way, but the function p no longer has to be a polynomial in ξ. Instead, $p(x,\xi)$ must belong to a certain *symbol class*. The following symbol classes were originally defined by Hörmander:

Definition 4.1. For $\rho,\ \delta\in[0,1]$ and $m\in\mathbb{R}$, the symbol class $S^m_{\rho,\delta}$ consists of all C^∞ functions $p(x,\xi)$ satisfying

$$|D_x^\beta D_\xi^\alpha p(x,\xi)|\le C_{\alpha\beta}\langle\xi\rangle^{m-\rho|\alpha|+\delta|\beta|}$$

for all α and β, where $\langle\xi\rangle=(1+|\xi|^2)^{1/2}$. In this case, we say that the operator $p(x,D)$ defined by (1) is in $OPS^m_{\rho,\delta}$. $p(x,\xi)$ is said to be the *symbol* of $p(x,D)$.

Example 4.1. The differential operator from before has a symbol of the form

$$p(x,\xi)=\sum_{|\alpha|\le k}a_\alpha(x)\xi^\alpha$$

Upon inspection, we can see that

$$|p(x,\xi)|\le C\langle\xi\rangle^k$$

because the highest power present is k. It is clear that derivatives in x have no impact on the growth of the function, since all of the coefficients a_α are smooth, and so all derivatives are bounded on the region of interest, and derivatives in ξ just reduce the order of growth by 1 power of ξ. Therefore, we have that operators of this type are in $S^k_{1,0}$. Therefore, operators in the classes of the form $S^m_{1,0}$ are of particular interest, because operators in this class should behave somewhat like normal differential operators.

4.2 Schwartz Kernels of Pseudodifferential Operators

A major result in the area of pseudodifferential operators is the fact that they can be represented by integral kernels.

Theorem 4.1 (Schwartz Kernels)**.** *Let $p(x,D)$ be an operator in $OPS^m_{\rho,\delta}$. Then, there exists a $K\in C_0^\infty(\mathbb{R}^n\times\mathbb{R}^n)^*$ so that*

$$\langle K,u(x)v(y)\rangle_{L^2(\mathbb{R}^{2n})}=\langle u,p(x,D)v\rangle_{L^2(\mathbb{R}^n)}$$

for all u and v in $C_0^\infty(\mathbb{R}^n)$.

Proof. See [5] for the full proof. The outline of the proof is as follows:

1. The operator $p(x, D)$ gives rise to a separately continuous bilinear form on $C^\infty(\Omega) \times C^\infty(\Omega)$ in the form
$$B[u, v] = \langle u, p(x, D)v\rangle$$
2. Since the form is separately continuous, there exist k and l positive numbers so that
$$|B[u, v]| \leq C||u||_{H^k(\Omega)}||v||_{H^l(\Omega)}$$
This is done via the Baire Category Theorem because the sequence of norms $|| \cdot ||_{H^j}$ characterizes C^∞.
3. This fact tells us that $p(x, D)$ maps H^k into H^{-l}, which we can pre- and post-compose with the isomorphisms
$$\Lambda^s := H^\sigma(\Omega) \to H^{\sigma-s}(\Omega) \qquad \widehat{\Lambda^s u} = (1 + |\xi|^2)^{s/2}\hat{u}$$
to write $p(x, D)$ as an operator from L^2 to L^2.
4. With the proper choice of exponents, $p(x, D)$ then becomes a Hilbert-Schmidt operator, which has an integral kernel $K(x, y)$.
5. Use integration by parts to remove the pre- and post-composed factors to get the kernel K of the pseudodifferential operator $p(x, D)$.

■

In order to get a handle on what this K looks like, we unpack the definition

$$\begin{aligned}\langle u, p(x, D)v\rangle_{L^2(\mathbb{R}^n)} &= (2\pi)^{-n} \iint u(x)p(x, \xi)\hat{v}(\xi)e^{ix\cdot\xi}\, d\xi\, dx \\ &= (2\pi)^{-n} \iiint u(x)p(x, \xi)v(y)e^{i(x-y)\cdot\xi}\, d\xi\, dx\, dy \\ &= \langle K, u(x)v(y)\rangle\end{aligned}$$

Therefore, the function K can be represented as

$$K(x, y) = (2\pi)^{-n} \int p(x, \xi)e^{i(x-y)\cdot\xi}\, d\xi \tag{2}$$

From this, we can prove some easy results on the regularity of this function.

Proposition 4.1. *If $\rho > 0$, then K is C^∞ off of the diagonal in $\mathbb{R}^n \times \mathbb{R}^n$.*

Proof. For any $\alpha \geq 0$, we have that

$$(x - y)^\alpha K = (2\pi)^{-n} \int e^{i(x-y)\cdot\xi} D_\xi^\alpha p(x, \xi)\, d\xi \tag{3}$$

Since $p \in OPS^m_{\rho,\delta}$ we know that
$$|D_\xi^\alpha p(x, \xi)| \leq C_{\alpha\beta}\langle\xi\rangle^{m-\rho|\alpha|}$$
which means that $|D_\xi^\alpha p(x, \xi)|$ is $L^1(\mathbb{R}^n)$ if $m - \rho|\alpha| < -n$. Therefore, by the properties of Fourier Transform discussed earlier, $(x - y)^\alpha K$ is continuous, so that K is continuous off the diagonal in $\mathbb{R}^n \times \mathbb{R}^n$. Similarly, applying j derivatives to (3) will give j extra powers of ξ, meaning we just need to choose α large enough so that $m + j - \rho|\alpha| < -n$ in order to conclude that $(x - y)^\alpha K \in C^j(\mathbb{R}^n \times \mathbb{R}^n)$, and thus K is C^j off the diagonal. Since this holds for all j, K is smooth off the diagonal. ■

Corollary 4.1. *If $p \in OPS^m_{\rho,\delta}$ with $\rho > 0$ and $m < -n$, then K is continuous on all of $\mathbb{R}^n \times \mathbb{R}^n$ and smooth off the diagonal.*

Proof. We follow the same proof as the proposition above. However, if $m < -n$, then the first part of the proof will go through with $\alpha = 0$, so we get that K is continuous without needing to multiply by $(x-y)$. ■

Corollary 4.2 (Decay Estimate)**.** *If $p \in OPS^m_{\rho,\delta}$ with $\rho > 0$ and $k \geq 0$ so that $k > \frac{1}{\rho}(m+n)$, then*

$$|K(x,y)| \leq C|x-y|^{-k}$$

Proof. By the same argument above, for such a k, we have that $(x-y)^\alpha K$ is defined by an absolutely convergent integral for $|\alpha| = k$. Therefore $(x-y)^\alpha K$ is bounded, and so the desired bound holds. ■

These kernels can also be written in the form

$$K(x,y) = L(x, x-y)$$

where the function

$$L(x,z) = (2\pi)^{-n} \int p(x,\xi) e^{iz\cdot\xi}\, d\xi$$

and by the results before, L is smooth on $\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$ and may be continuous on the whole space depending on the order m of the operator. This form of the kernel will be more useful later.

5 Gårding's Asymptotics

Now, we want to bring all of this theory together to talk about Gårding's result. We already know that for t large enough, the operator $a_t := a + t$ is invertible with inverse G_t. In order to prove his result, Gårding seeks to find an integral kernel for this operator G_t so that he can connect the eigenvalues of a to the trace of the operator. To follow his argument, we will also find a kernel function, but ours will come from the theory of pseudodifferential operators.

Consider a, our original elliptic differential operator of order $2m$, which can be represented by a pseudodifferential operator with symbol

$$a(x,\xi) = \sum_{|\alpha| \leq 2m} a_\alpha(x)\xi^\alpha$$

so that the symbol of the invertible operator $a_t = a + t$ is given by

$$a_t(x,\xi) = \sum_{|\alpha| \leq 2m} a_\alpha(x)\xi^\alpha + t$$

Then, we can set $t = \tau^{2m}$ and define

$$b_0^\tau(x,\xi) = \frac{1}{\tau^{2m}} a_t(x,\tau\xi) = \sum_{|\alpha| \leq 2m} a_\alpha(x)\tau^{|\alpha|-2m}\xi^\alpha + 1$$

so that

$$b_0^\infty(x,\xi) = \sum_{|\alpha| = 2m} a_\alpha(x)\xi^\alpha + 1 = a_0(x,\xi) + 1$$

and we define

$$p_\tau(x,\xi) := \frac{1}{b_0^\tau(x,\xi)}$$

Since $a_0(x,\xi) \geq C|\xi|^{2m}$ by ellipticity, we then have that

$$p_\infty(x,\xi) := \frac{1}{b_0^\infty(x,\xi)} \leq C(1+|\xi|^2)^{-m} = C\langle\xi\rangle^{-2m}$$

Finally, since b_0^∞ is a polynomial in ξ, the derivatives in ξ of $p_\infty(x,\xi)$ will each lower the order by 1. Therefore, we have that

$$p_\infty(x,\xi) \in S^{-2m}_{1,0}$$

as a pseudodifferential operator. Since all of these bounds are continuous in τ, the fact that $p_\infty \in S^{-2m}_{1,0}$ implies that $p_\tau \in S^{-2m}_{1,0}$ for τ large enough. Therefore, for each τ, p_τ has an integral kernel L_τ so that

$$\langle u, p_\tau(x,D)v\rangle = \iint L_\tau(x,x-y)u(x)v(y)\ dx\ dy$$

where

$$L_\tau(x,z) = (2\pi)^{-n}\int p_\tau(x,\xi)e^{iz\cdot\xi}\ d\xi$$

This is similar to the B function that Gårding forms in his paper, with a shift to put the pole at 0. With this setup, we have that L_τ is an integral kernel for the inverse operator to $b_0^\tau(x,D)$ because

$$\begin{aligned}\iint L_\tau(x,x-y)b_0^\tau(x,D)u(x)v(y)\ dx\ dy &= \int b_0^\tau(x,D)u(x)p_\tau(x,D)v(x)\ dx\\ &= \int b_0^\tau(x,\xi)\hat{u}(\xi)p_\tau(x,\xi)\hat{v}(\xi)\ d\xi \\ &= \int \hat{u}(\xi)\hat{v}(\xi) = (u,v)_{L^2(\mathbb{R}^n)}\end{aligned} \tag{4}$$

where the last line follows by the definition of p_τ. In the same way, we would like to find an inverse to the operator $a_t(x,D)$. To this end, we define the function h_t by

$$h_t(x,z) = \tau^{n-2m}L_\tau(x,\tau z) \qquad t = \tau^{2m}$$

and $g_t(x,y) = h_t(x,x-y)$. In the same sense that L_τ is the integral kernel for the operator p_τ, we see that by change of variables

$$\begin{aligned}h_t(x,z) = \tau^{n-2m}L_\tau(x,\tau z) &= (2\pi)^{-n}\tau^{n-2m}\int p_\tau(x,\xi)e^{\tau z\cdot\xi}\ d\xi\\ &= (2\pi)^{-n}\tau^{n-2m}\int p_\tau(x,\frac{\tau\xi}{\tau})e^{z\cdot\tau\xi}\ d\xi\\ &= (2\pi)^{-n}\int \frac{1}{\tau^{2m}}p_\tau\left(x,\frac{\eta}{\tau}\right)e^{iz\cdot\eta}\ d\eta\end{aligned}$$

that h_t is the integral kernel corresponding the operator with symbol $\tau^{-2m}p_\tau(x,\xi/\tau)$. Using the definition of p_τ, we then have that

$$\tau^{-2m}p_\tau(x,\xi/\tau) = \tau^{-2m}\frac{1}{b_0^\tau(x,\xi/\tau)} = \frac{1}{a_t(x,\xi)}$$

Therefore, the exact same steps as in (4) give that

$$\iint g_t(x,y)a_t(x,D)u(x)v(y)\ dx\ dy = \iint h_t(x,x-y)a_t(x,D)u(x)v(y)\ dx\ dy = (u,v)_{L^2(\mathbb{R}^n)}$$

so that this g_t is an integral kernel for the inverse operator G_t to the differential operator a_t.Therefore, this g_t is the integral kernel for the Green's operator G_t that Gårding derives. Now, we just need to prove the asymptotic part of the result.

If $2m > n$, then from the work on pseudodifferential operators we know that the kernel L_τ is continuous on all of $\mathbb{R}^n \times \mathbb{R}^n$, which implies that g_t is continuous. If $2m < n$, then we can consider powers of the operator a_t, so that a_t^k will have order $2mk$, and choose a k large enough to make this larger than n. In this case, we define

$$b_0^\tau(x,\xi) = \frac{1}{\tau^{2mk}}(a_t(x,\tau\xi))^k$$

and the rest of the definitions are adjusted accordingly. Once we have this, the corresponding Green's operator G_t^k will have continuous integral kernel $g_t^{(k)}$. In the case that these kernels are continuous, we can evaluate both L_τ and g_t everywhere, including along the diagonal. Doing this, we see that

$$L_\tau(x,0) = (2\pi)^{-n}\int p_\tau(x,\xi)\ d\xi$$

so that

$$\lim_{\tau\to\infty} L_\tau(x,0) = (2\pi)^{-n}\int p_\infty(x,\xi)\ d\xi = (2\pi)^{-n}\int (a_0(x,\xi)+1)^{-k}\ d\xi$$

and similarly, we have, for $\nu = n/2m$

$$\begin{aligned}\lim_{t\to\infty} t^{k-\nu} g_t^k(x,x) &= \lim_{t\to\infty} t^{k-\nu} h_t^k(x,0) = \lim_{\tau\to\infty} \tau^{2mk-n}\tau^{n-2mk} L_\tau(x,0)\\ &= (2\pi)^{-n}\int p_\infty(x,\xi)\ d\xi = (2\pi)^{-n}\int (a_0(x,\xi)+1)^{-k}\ d\xi\end{aligned}$$

Furthermore, if $x \neq y$, then

$$\lim_{t\to\infty} t^{k-\nu} g_t^k(x,y) = \lim_{t\to\infty} t^{k-\nu} h_t^k(x,x-y) = \lim_{\tau\to\infty} L_\tau(x,\tau(x-y)) = 0$$

by the decay estimate, which we can concisely write as

$$\lim_{t\to\infty} t^{k-\nu} g_t^{(k)}(x,y) = \delta_{xy}(2\pi)^{-n}\int (a_0(x,\xi)+1)^{-k}\ d\xi$$

which is the same expression that Gårding gets to in his paper for the kernel.

Now, we want to restrict to the diagonal terms and integrate this expression over Ω to get that

$$\lim_{t\to\infty} t^{k-\nu}\int_\Omega g_t^{(k)}(x,x)\ dx = (2\pi)^{-n}\int_\Omega\int_{\mathbb{R}^n} (a_0(x,\xi)+1)^{-k}\ d\xi\ dx \tag{5}$$

which holds by Dominated Convergence because for τ large enough,

$$t^{k-\nu} g_t^{(k)}(x,x) = L_\tau(x,0) = (2\pi)^{-n}\int p_\tau(x,\xi)\ d\xi$$

and p_τ is bounded by $C\langle\xi\rangle^{-2km}$ which is L^1.

Since G_t is self-adjoint and positive, there exists a complete L^2-orthonormal system of eigenfunctions ϕ_j for G_t with positive eigenvalues

$$(\lambda_1+t)^{-1} \geq (\lambda_2+t)^{-1} \geq \cdots$$

From this, we have that

$$a_t(\phi_j,f) = (\lambda_j+t)a_t(G_t\phi_j,f) = (\lambda+t)(\phi_j,f)$$

for any $f \in H_0^m(D)$ Since, for any other $s \in \mathbb{R}$,

$$a_s(f,g) = a(f,g) + s(f,g) = a_t(f,g) + (s-t)(f,g)$$

we can see that if ϕ_j is an eigenfunction for G_t, it is also an eigenfunction for G_s for all other s. Thus, we have that these ϕ_j are also eigenfunctions for a in the sense that

$$a\phi_j = \lambda_j \phi_j$$

Next, we need to show that the kernel function $g_t^{(k)}(x,y)$ can be decomposed as

$$g_t^{(k)}(x,y) = \sum_j (\lambda_j + t)^{-k} \overline{\phi_j(x)} \phi_j(y)$$

as long as $2mk > n$ so that

$$\int_\Omega g_t^{(k)}(x,x)\ dx = \sum (\lambda_j + t)^{-k} = trG_t^k \tag{6}$$

We know that $\int_S |g_t^{(k)}(x,y)|^2\ dy < \infty$ because g_t^k is continuous on the diagonal and smooth off it, with decay for $|x-y|$ large. Therefore, by Fubini's theorem we have that

$$(\lambda_j + t)^{-k}(f, \phi_j) = (G_t^k f, \phi_j) = \int \left[\int g_t^{(k)}(x,z) f(x)\ dx \right] \overline{\phi_j(z)}\ dz = \int \left[\int g_t^{(k)}(x,z) \overline{\phi_j(z)}\ dz \right] f(x)\ dx$$

for any $f \in C_0^\infty(S)$, so that

$$\int g_t^{(k)}(x,z) \overline{\phi_j(z)}\ dz = (\lambda_j + t)^{-k} \overline{\phi_j(x)}$$

Since $\{\phi_j\}$ is a complete orthonormal basis for L^2, we can write

$$g_t^{(k)}(x,z) = \sum_n c_n(x) \phi_j(z)$$

and the above relation implies that

$$c_n(x) = (\lambda_j + t)^{-k} \overline{\phi_j(x)}$$

and so $g_t^{(k)}$ has the desired form

$$g_t^{(k)}(x,y) = \sum (\lambda_j + t)^{-k} \overline{\phi_j(x)} \phi_j(y)$$

Plugging our previous estimates into (6) gives that

$$\lim_{t\to\infty} t^{k-\nu} \sum (\lambda_j + t)^{-k} = (2\pi)^{-n} \int_\Omega \int_{\mathbb{R}^n} (a_0(x,\xi) + 1)^{-k}\ d\xi\ dx$$

which gives an asymptotic relation on the sum of the eigenvalues, namely that

$$\sum (\lambda_j + t)^{-k} \sim \frac{H}{t^{k-\nu}}$$

where

$$H := (2\pi)^{-n} \int_S \int_{\mathbb{R}^n} (a_0(x,\xi) + 1)^{-k}\ d\xi\ dx$$

Then, using Theorem 3.3 from earlier, we get that

$$N(x) \sim H \frac{\Gamma(k)}{\Gamma(k-\nu)\Gamma(\nu+1)} x^\nu = H' x^{n/2m}$$

which is the exact order we expect to see. Now, we need to calculate this H and H'.

If we define

$$w_a^{(k)}(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} (a_0(x,\xi)+1)^{-k} \ d\xi$$

then if ρ, ω are polar coordinates on $\mathbb{R}^n$ with $\rho^{2m} = a_0(x,\xi)$, then

$$w_a^{(k)}(x) = (2\pi)^{-n} \iint_{\mathbb{R}^n} (\rho^{2m}+1)^{-k} \ d\rho^n \ d\omega = (2\pi)^{-n} \int d\omega \int_0^\infty (\rho^{2m}+1)^{-k} \ d\rho^n$$

and we can compute the surface integral as

$$\int d\omega = \int \int_0^1 d\rho^n d\omega = \int_{a_0(x,\xi)<1} d\xi$$

For the radial direction, we see that

$$\int_0^\infty (\rho^{2m}+1)^{-k} \ d\rho^n = n \int_0^\infty \rho^{n-1} (\rho^{2m}+1)^{-k} \ d\rho$$

Now, we make the change of variables $t = (\rho^{2m}+1)^{-1}$, noting that

$$\rho = \left(\frac{1-t}{t}\right)^{1/2m} \qquad dt = -2m(\rho^{2m}+1)^{-2}\rho^{2m-1} \ d\rho = -2mt^2 \left(\frac{1-t}{t}\right)^{\frac{2m-1}{2m}} \ d\rho$$

to get

$$\begin{aligned}
\int_0^\infty (\rho^{2m}+1)^{-k} \ d\rho^n &= \frac{n}{2m} \int_0^1 \left(\frac{1-t}{t}\right)^{\frac{n-1}{2m}} t^k \frac{1}{t^2} \left(\frac{1-t}{t}\right)^{\frac{1-2m}{2m}} \ dt \\
&= \nu \int_0^1 (1-t)^{\nu-1} t^{k-\nu-1} \ dt \\
&= \nu B(\nu, k-\nu) = \nu \frac{\Gamma(\nu)\Gamma(k-\nu)}{\Gamma(k)} = \frac{\Gamma(\nu+1)\Gamma(k-\nu)}{\Gamma(k)}
\end{aligned}$$

where $B(a,b)$ is the Euler Beta function. Therefore, our constant H is

$$H := (2\pi)^{-n} \int_S \int_{\mathbb{R}^n} (a_0(x,\xi)+1)^{-k} \ d\xi \ dx = (2\pi)^{-n} \frac{\Gamma(\nu+1)\Gamma(k-\nu)}{\Gamma(k)} \iint_{a_0(x,\xi)<1} d\xi \ dx$$

and so the coefficient H' on the asymptotics of the eigenvalues is just

$$H' := (2\pi)^{-n} \iint_{a_0(x,\xi)<1} d\xi \ dx$$

which gives us

$$N(\lambda) \sim (2\pi)^{-n} \lambda^{n/2m} \iint_{a_0(x,\xi)<1} d\xi \ dx$$

as the final desired result from [2].

Remark. This result can also be written in the form

$$N(\lambda) \sim (2\pi)^{-n} \iint_{a_0(x,\xi)<\lambda} d\xi \ dx$$

because we are in n dimensional space, and a_0 is homogeneous in ξ of degree $2m$.

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