Thermal Detection of Inaccessible Plate Corrosion

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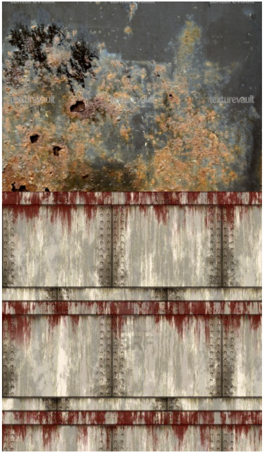
Outline

Motivation



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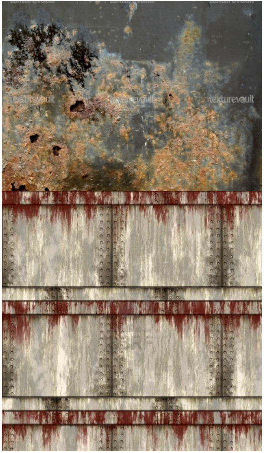

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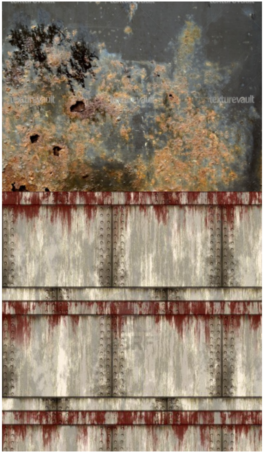

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- Finding possible corrosion in a chemical pipe,

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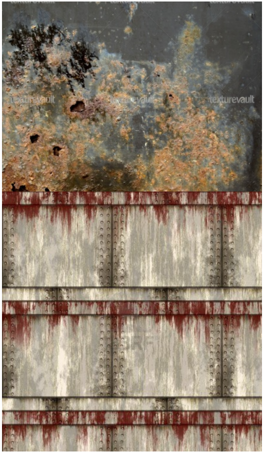

We want to be able to determine if an inaccessible face of a material has been corroded by applying heat and taking temperature measurements on an accesible face. Possible applications include:

- Detecting external hull corrosion on a vessel from the inside,
- Finding possible corrosion in a chemical pipe, or
- Testing ceramic materials for imperfections.

Why Use Heat?

The flow of heat is well understood, from both a mathematical and engineering perspective. Previous work has used electric fluxes to find corrosion. [1].

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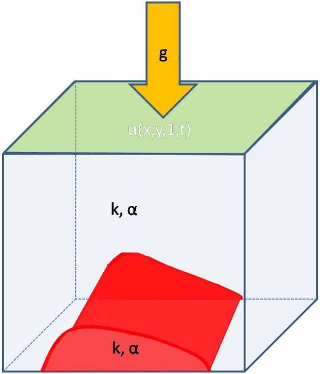
The flow of heat is well understood, from both a mathematical and engineering perspective. Previous work has used electric fluxes to find corrosion. [1].

However, heat has many advantages over electricity.

- Completely non-contact method
- Time dependence may be able to reveal more information

Full Problem



Given:

- A thermally conductive object, with thermal conductivity and diffusivity.
- A region of the object that is corroded, with different properties.
- A heat flux $g(x, y)$ applied to the top surface.

Determine $u(x, y, z, t)$, the temperature in the block.

The Heat Equation

Under some basic physics/engineering assumptions, we know that the temperature u in the body must satisfy

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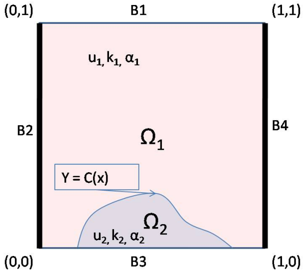
This can be derived from the conservation of energy and describes how heat travels through a medium.

Simplified Problem

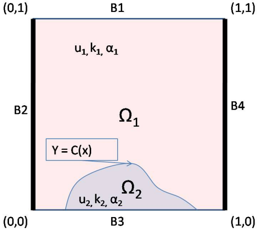
Given the unit square with:

- Two regions, Ω_1 and Ω_2, separated by a curve $C(x)$.
- The thermal properties of both Ω_1 and Ω_2.
- A heat flux $g(x)$ applied to the top surface $(y = 1)$.

Determine the temperature in the block $u(x, y, t)$ at all times.



Formal Problem Statement

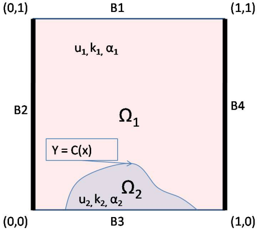


After rescaling the problem to the unit square, we know that u_1 satisfies:

$$\begin{aligned}
\frac{\partial u_1}{\partial t} - \alpha_1 \nabla^2 u_1 &= 0 \text{ on } \Omega_1 \\
\frac{\partial u_1}{\partial x} &= 0 \text{ on } B2, \ B4 \\
\frac{\partial u_1}{\partial y} &= g(x) \text{ on } B1 \\
u_1(x, y, 0) &= 0 \text{ on } \Omega_1
\end{aligned}$$

Formal Problem Statement

Similarly, u_2 satisfies:



$$
\begin{aligned}
\frac{\partial u_2}{\partial t} - \alpha_2 \nabla^2 u_2 &= 0 \text{ on } \Omega_2 \\
\frac{\partial u_2}{\partial x} &= 0 \text{ on } B2, \ B4 \\
\frac{\partial u_2}{\partial y} &= 0 \text{ on } B3 \\
u_2(x, y, 0) &= 0 \text{ on } \Omega_2
\end{aligned}
$$

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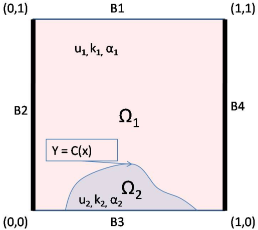

The two solutions together also satisfy continuity conditions on $C(x)$:

$$\begin{aligned} u_1 &= u_2 \text{ on } C(x) \\ k_1 \frac{\partial u_1}{\partial \vec{n}} &= k_2 \frac{\partial u_2}{\partial \vec{n}} \text{ on } C(x) \end{aligned}$$

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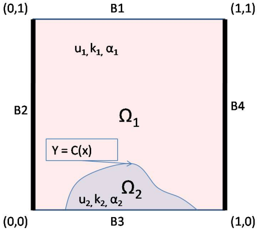

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Given these equations, the flux $g(x)$, and curve $C(x)$ that bounds the corroded region Ω_2, can we find the functions $u_1(x, y, t)$ and $u_2(x, y, t)$ satisfying all of them?

Solutions to the Heat Equation

The Heat Equation can be solved analytically by many methods, including Fourier series and Green's functions

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In cases where analytical methods fail, numerical solutions are avaiable.

Ex: Spread of a point heat impulse.

Inverse Problems

What we just saw was the "Forward Problem"; given a governing equation, initial and boundary conditions, find a function that satisfies all of these for our solution.

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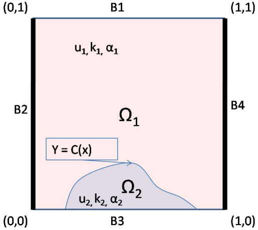
This leads to a mathematical Inverse Problem.

Inverse Problem Statement



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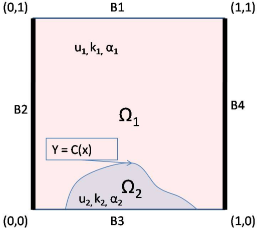

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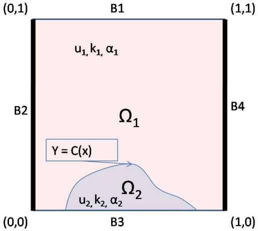

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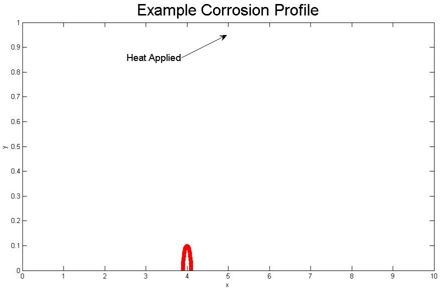
We are given $u_1(x, 1, t)$ for all x, t corresponding to the value of the solution on the top surface.

Given this portion of the solution, and the other constraints on the problem, can we now determine $C(x)$?

Corrosion Example

Let's take a look at the value of the solution on the top boundary in the corroded vs. uncorroded case. Below is an example of a small corrosion profile.



Corrosion Example

Comparing our solutions:

Surface temperatures for corroded versus uncorroded plates

Difference between corroded and uncorroded.

Uniqueness

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- Use a Uniqueness theorem by O. Poisson [3].

While we have a uniqueness result, we haven't yet defined a method to extract our curve $C(x)$ from the partial solution data. We move to this next.

Mathematical Approach

The mathematical approach to this inverse problem involves three steps:

- Linearization
- Integration with Green's Identity
- Regularization

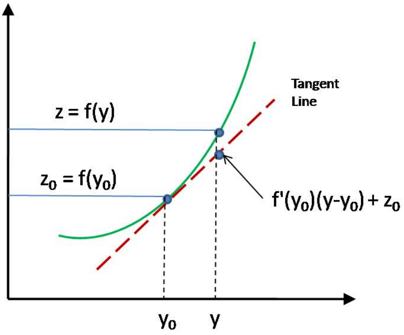
Linearization: Single Variable

Let $z = f(y)$ be a continuous and differentiable function on some open interval.

Then, starting at some point y_0, we can approximate the value of $z = f(y)$ by

$$z \approx z_0 + f'(y_0)(y - y_0)$$

for y sufficiently close to y_0.

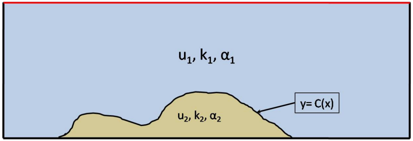


Linearization: Multiple Variables

Assume that u_1, u_2, and $C(x)$ are small perturbations of the uncorroded situation:

$$\begin{aligned} u_1 &= u_0 + \tilde{u}_1 = u_0 + \epsilon \bar{u}_1 \\ u_2 &= u_0 + \tilde{u}_2 = u_0 + \epsilon \bar{u}_2 \\ C(x) &= 0 + \epsilon C_0(x) \end{aligned}$$

We now look at the perturbation in region 1, $\tilde{u}_1$.



Linearization: Multiple Variables

We know $\tilde{u}_1$ satisfies

$$\begin{aligned}
\frac{\partial \tilde{u}_1}{\partial t} - \alpha_1 \nabla^2 \tilde{u}_1 &= 0 \quad \text{on } \Omega \\
\frac{\partial \tilde{u}_1}{\partial \vec{n}} &= 0 \quad \text{on sides and top} \\
\tilde{u}_1(x,y,0) &= 0
\end{aligned}$$

as well as the flux continuity condition on $C(x)$

$$k_1 \frac{\partial \tilde{u}_1}{\partial \vec{n}} = (k_2 - k_1)\frac{\partial u_0}{\partial \vec{n}} + k_2 \frac{\partial \tilde{u}_2}{\partial \vec{n}} \tag{1}$$

Linearization: Multiple Variables

Performing a linearization procedure on all three terms in Equation 1 gives the following expression for the normal derivative of $\tilde{u}_1$:

$$\frac{\partial \tilde{u}_1}{\partial \vec{y}}|_{y=0} = \frac{k_1 - k_2}{k_1}\left(C(x)\frac{\partial^2 u_0}{\partial y^2}|_{y=0} - C'(x)\frac{\partial u_0}{\partial x}|_{y=0}\right)$$

or

$$\frac{\partial \tilde{u}_1}{\partial \vec{y}}|_{y=0} = \frac{k_1 - k_2}{k_1}\left(\frac{C(x)}{\alpha_1}\frac{\partial u_0}{\partial t}|_{y=0} - \frac{\partial}{\partial x}\left(C(x)\frac{\partial u_0}{\partial x}|_{y=0}\right)\right) \quad (2)$$

Green's Second Identity

Now that we have a linearized problem, Green's Second Identity, which follows from the Divergence Theorem, will allow us to use what we know, the temperature on the top surface, to approximate the curve $C(x)$.

Theorem (Green's Second Identity)

Let $D \subset \mathbb{R}^2$ be a simply connected region in the plane, and let ∂D be the boundary of D. Then, for any $u, v \in C^2(\mathbb{R}^2)$,

$$\int_D u\nabla^2 v - v\nabla^2 u \; dA = \int_{\partial D} u\frac{\partial v}{\partial \vec{n}} - v\frac{\partial u}{\partial \vec{n}} \; ds$$

Test Functions

We define a test function ϕ such that:

$$
\begin{aligned}
\frac{\partial \phi}{\partial t} + \alpha_1 \nabla^2 \phi &= 0 \text{ on } \Omega \\
\frac{\partial \phi}{\partial x} &= 0 \text{ on sides} \\
\frac{\partial \phi}{\partial y} &= 0 \text{ on bottom} \\
\phi(x, y, T) &= 0 \text{ on } \Omega
\end{aligned}
$$

These test functions are generated numerically using the Green's Function for heat and the method of images to give the desired zero Neumann data conditions.

Green's Identity Integration

$$\int_0^T \int_\Omega \phi \left(\frac{\partial \tilde{u}_1}{\partial t} - \alpha \nabla^2 \tilde{u}_1 \right) \, dA \, dt = 0$$

Using Green's Identity, integration by parts, and Equation 2 gives:

$$RG(\phi_k) := \int_0^T \int_{\text{top}} \tilde{u}_1 \frac{\partial \phi_k}{\partial \vec{n}} \, ds \, dt = \int_0^l C(x) \underbrace{\int_0^T \frac{\partial \phi_k}{\partial x} \frac{\partial u_0}{\partial x} - \frac{u_0}{\alpha_1} \frac{\partial \phi_k}{\partial t} \, dt}_{w_k(x)} \, dx \qquad (3)$$

where $RG(\phi_k)$ is known completely from the collected data and choice of ϕ_k.

Problem Solution

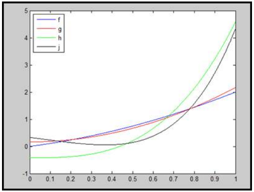
$$RG(\phi_k) = \int_0^l C(x) w_k(x)\ dx$$

There are many functions $C(x)$ that will solve this problem.

Assume: $\int_0^1 C(x)\ dx = \frac{5}{6}$ and $\int_0^1 xC(x)\ dx = \frac{7}{12}$

There are infinitely many functions that satisfy these constraints:

$$\begin{aligned} f(x) &= x^2 + x \\ g(x) &= 2x^2 + \frac{1}{6} \\ h(x) &= 5x^3 - \frac{5}{12} \\ j(x) &= 5x^4 - x + \frac{1}{3} \end{aligned}$$



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There are many functions $C(x)$ that will solve this problem.

In order to specify a single function, we look for one with the smallest L^2 norm. It can be shown that such a function is of the form

$$C(x) = \sum_{i=1}^{N} \lambda_i w_i(x)$$

Problem Solution

$$RG(\phi_k) = \int_0^l \sum_{i=1}^{N} \lambda_i w_i(x) w_k(x)\ dx$$

which can be rewritten as:

$$RG(\phi_k) = \sum_{i=1}^{N} \lambda_i \int_0^l w_i(x) w_k(x)\ dx$$

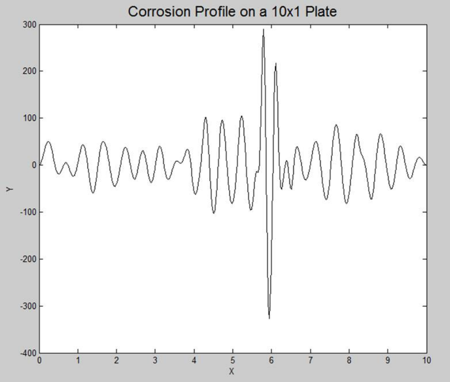
Defining $\mathbf{B}$ such that

$$\mathbf{B}_{ij} = \int_0^l w_i(x) w_j(x)\ dx$$

gives

$$\vec{RG} = \mathbf{B}\vec{\lambda} \quad \Rightarrow \quad \vec{\lambda} = \mathbf{B}^{-1}\vec{RG}$$

Problem Solution



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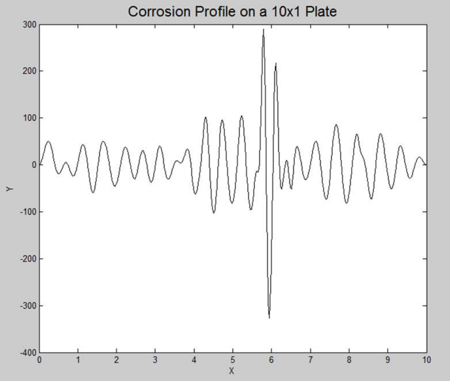


This is obviously a problem.

Regularization: Ill-posedness

This inverse problem is very ill-posed/ill-conditioned. The matrix $\mathbf{B}$ has singular values very close to zero, so $\mathbf{B}^{-1}$ has very large values. This results in large λ_i values, causing the approximation of $C(x)$ to be well beyond physical constraints.

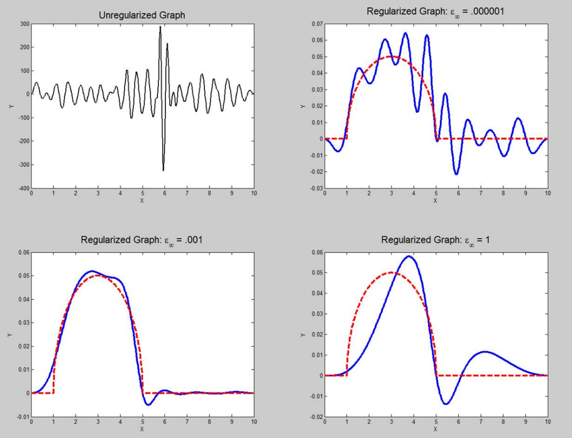
In order to produce a feasible solution, we need to regularize the problem.

Regularization Methods

Our regularization method involves the Singular Value Decomposition of the matrix $\mathbf{B}$.

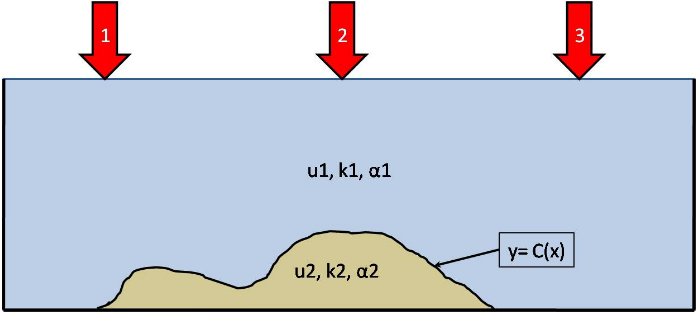
We want to eliminate the extremely large values in $\mathbf{B}^{-1}$. To do this, we look at the singular values of $\mathbf{B}$. If the value is under a certain threshold, the corresponding value in $\mathbf{B}^{-1}$ is set to zero, removing the high amplitude values from $\vec{\lambda}$.

Regularization Methods



Method of Computing Results

Three separate trials to avoid errors near the laser source.



Results: Single Corrosion Profile

Note: In all of the following reconstructions, we used

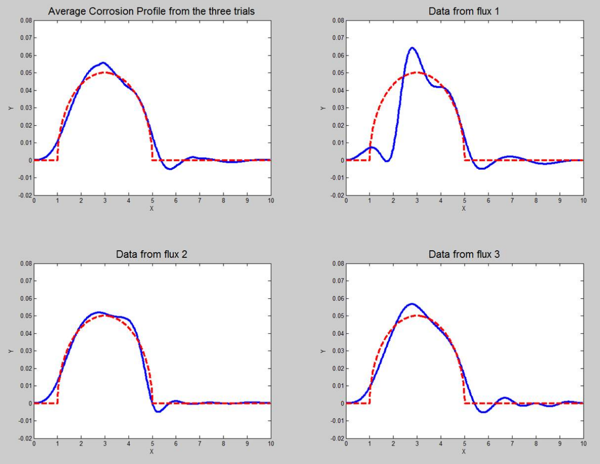
$$w_k(x) = \int_0^T \frac{k_2 - k_1}{k_1} \frac{\partial u_0}{\partial x} \frac{\partial \phi_k}{\partial x} \ dt$$

instead of Equation 3 because this gave significantly improved results.

Results: Single Corrosion Profile

Actual Corrosion Profile

Estimated Corrosion Profile



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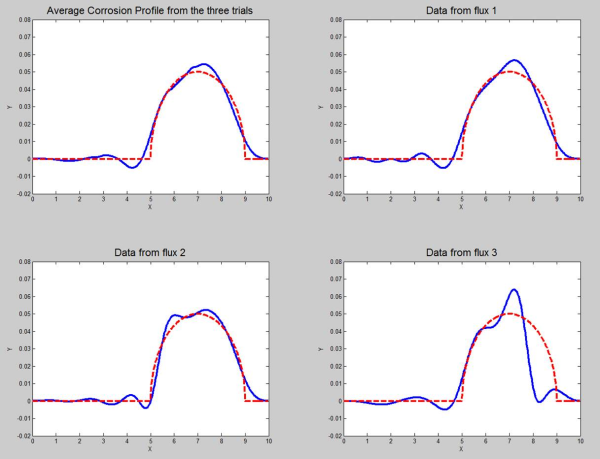
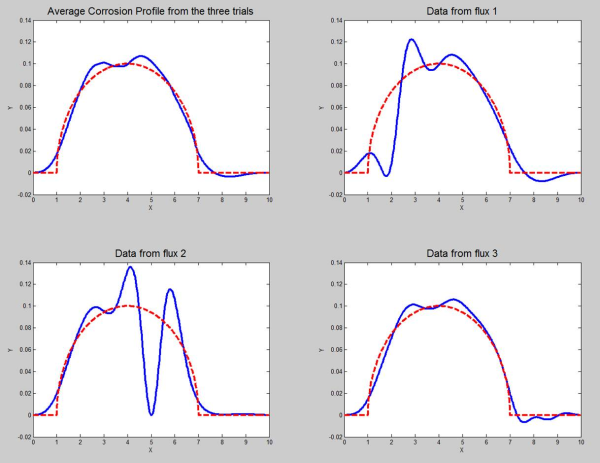


Results: Large Corrosion Profile

Actual Corrosion Profile

Estimated Corrosion Profile



Results: Multiple Corrosion Profiles

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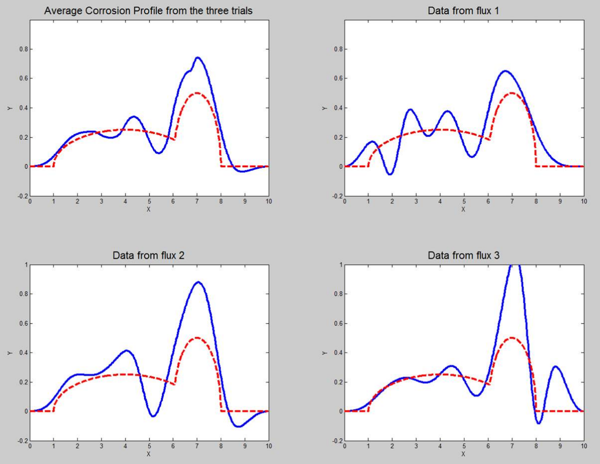

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Results: Discussion

- We can find corrosion!!
- We can definitely see where it is, and can estimate the area of the corrosion within an acceptable error.
- Even for large corrosion, which is well outside the range of linearization, we can see where the majority of the corrosion is.
- Reconstruction only takes about a minute after the test functions have been computed.

Future Work

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2. Use parameters from actual metals and equipment.
3. Other ways of utilizing the time dependence of the heat equation.
4. Time-Dependent Fluxes
5. Full 3-Dimensional problem

Questions?

References

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Electrical Impedance Imaging of Corrosion on a Partially Accessible 2-Dimensional Region: An Inverse Problem in Non-destructive Testing
Rose-Hulman REU 2009

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Reconstuction of an unknown boundary portion from Cauchy data in n dimensions.
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[3] Olivier Poisson
Uniqueness and Holder stability of discontinuous diffusion coefficients in three related inverse problems for the heat equation
IOP Publishing, Inverse Problems **24** (2008) 025012

Linearization

Consider:

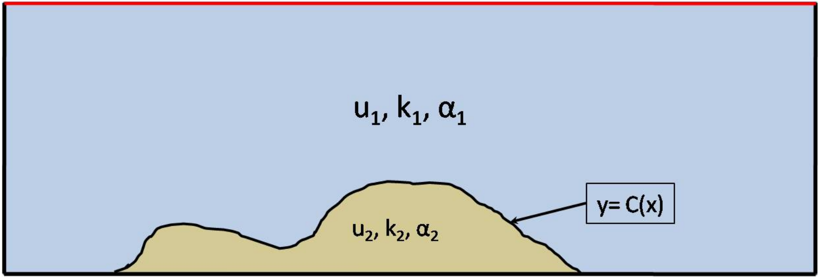
$$M : C(x) \to D$$

the map from corrosion profiles $C(x)$ to the temperature data D on the top surface.

This map is very non-linear and can be computed via a forward problem/heat equation solver. Linearizing the problem will allow for an inverse map to be approximated.

Linearization: Multiple Variables

In our case, we have a function of multiple variables, but for each (x, t), we can consider u as only a function of y, and linearize it with respect to that variable.



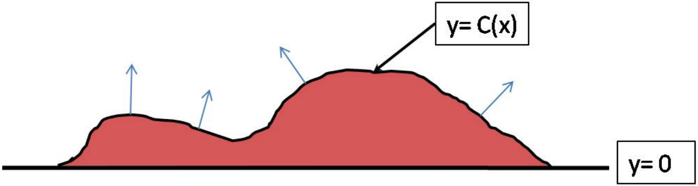
Linearization: Multiple Variables

Since

$$\vec{n} = \frac{< -C'(x), 1 >}{\sqrt{C'(x)^2 + 1}}$$

We can write

$$k_1 \frac{\partial \tilde{u}_1}{\partial \vec{n}}|_{C(x)} = k_1 \left(\frac{\partial \tilde{u}_1}{\partial y}|_{C(x)} - C'(x) \frac{\partial \tilde{u}_1}{\partial x}|_{C(x)} \right)$$



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And, doing a linearization about 0 as discussed before, we get

$$k_1 \frac{\partial \tilde{u}_1}{\partial \vec{n}}|_{C(x)} = k_1 \left(\frac{\partial \tilde{u}_1}{\partial y}|_{y=0} + C(x) \frac{\partial^2 \tilde{u}_1}{\partial y^2}|_{y=0} - C'(x) \frac{\partial \tilde{u}_1}{\partial x}|_{y=0} - C(x)C'(x) \frac{\partial^2 \tilde{u}_1}{\partial x \partial y}|_{y=0} \right)$$

ignoring terms of higher orders.

Linearization: Multiple Variables

Similarly, for the other two terms:

$$
\begin{aligned}
& (k_2 - k_1)\frac{\partial u_0}{\partial \vec{n}} = (k_2 - k_1)\left(\frac{\partial u_0}{\partial y} - C'(x)\frac{\partial u_0}{\partial x}\right) \\
= \quad & (k_2 - k_1)\left(\frac{\partial u_0}{\partial y}|_{y=0} + C(x)\frac{\partial^2 u_0}{\partial y^2}|_{y=0} - C'(x)\frac{\partial u_0}{\partial x}|_{y=0} - C(x)C'(x)\frac{\partial^2 u_0}{\partial x \partial y}|_{y=0}\right) \\
& k_2\frac{\partial \tilde{u}_2}{\partial \vec{n}} = k_2\left(\frac{\partial \tilde{u}_2}{\partial y} - C'(x)\frac{\partial \tilde{u}_2}{\partial x}\right) \\
= \quad & k_2\left(\frac{\partial \tilde{u}_2}{\partial y}|_{y=0} + C(x)\frac{\partial^2 \tilde{u}_2}{\partial y^2}|_{y=0} - C'(x)\frac{\partial \tilde{u}_2}{\partial x}|_{y=0} - C(x)C'(x)\frac{\partial^2 \tilde{u}_2}{\partial x \partial y}|_{y=0}\right)
\end{aligned}
$$

Linearization: Multiple Variables

Substituting in, assuming that all of the perturbations are of order ϵ, gives

$$k_1 \frac{\partial \tilde{u}_1}{\partial \vec{n}} = (k_2 - k_1)\frac{\partial u_0}{\partial \vec{n}} + k_2 \frac{\partial \tilde{u}_2}{\partial \vec{n}}$$

$$k_1 \frac{\partial \tilde{u}_1}{\partial y}|_{y=0} = (k_2 - k_1)\left(\frac{\partial u_0}{\partial y} + C(x)\frac{\partial^2 u_0}{\partial y^2} - C'(x)\frac{\partial u_0}{\partial x}\right) + k_2 \frac{\partial \tilde{u}_2}{\partial y} + O(\epsilon^2)$$

with everything evaluated at $y = 0$. However, since

$$\frac{\partial u_0}{\partial y}|_{y=0} = \frac{\partial \tilde{u}_2}{\partial y}|_{y=0} = 0$$

we are left with

$$k_1 \frac{\partial \tilde{u}_1}{\partial y}|_{y=0} = (k_2 - k_1)\left(C(x)\frac{\partial^2 u_0}{\partial y^2}|_{y=0} - C'(x)\frac{\partial u_0}{\partial x}|_{y=0}\right)$$

Green's Integration

$$\int_0^T \int_\Omega \phi \left(\frac{\partial \tilde{u}_1}{\partial t} - \alpha \nabla^2 \tilde{u}_1 \right) \, dA \, dt = 0$$

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Integrating by parts and using Green's Identity gives

$$\int_0^T \int_{\text{top}} \tilde{u}_1 \frac{\partial \phi}{\partial \vec{n}} \, ds \, dt = \int_0^T \int_{\text{bottom}} \phi \frac{\partial \tilde{u}_1}{\partial \vec{n}} \, ds \, dt$$

since these functions are zero elsewhere on the boundary.

Green's Integration

$$\int_0^T \int_{\text{top}} \tilde{u}_1 \frac{\partial \phi}{\partial \vec{n}} \, ds \, dt = \int_0^T \int_{\text{bottom}} \phi \frac{\partial \tilde{u}_1}{\partial \vec{n}} \, ds \, dt$$

Since we collect data for $\tilde{u}_1$ on the top and know ϕ explicitly, the entire left side of this equation is known. This is defined as the Reciprocity Gap integral, or $RG(\phi)$.

Green's Integration

$$\int_0^T \int_{\text{top}} \tilde{u}_1 \frac{\partial \phi}{\partial \vec{n}} \, ds \, dt = \int_0^T \int_{\text{bottom}} \phi \frac{\partial \tilde{u}_1}{\partial \vec{n}} \, ds \, dt$$

Since we collect data for $\tilde{u}_1$ on the top and know ϕ explicitly, the entire left side of this equation is known. This is defined as the Reciprocity Gap integral, or $RG(\phi)$.

Using Equation 2 and integrating by parts, this equation can be simplified to

$$RG(\phi_k) = \int_0^l C(x) \underbrace{\int_0^T \frac{\partial \phi_k}{\partial x} \frac{\partial u_0}{\partial x} - \frac{u_0}{\alpha_1} \frac{\partial \phi_k}{\partial t} \, dt}_{w_k(x)} \, dx$$

Regularization Methods

One possible method of regularization is the *Tikhonov Regularization*, which for this problem would involve minimizing

$$\tilde{Q} = \sum_{k=1}^{N} \left| RG_k - \int_0^l C(x) w_k(x)\ dx \right| + \beta \left\| C(x) \right\|_2$$

where β is an adjustable regularization parameter.

Test Function Generation

We want to construct a set of test functions ϕ such that

$$\begin{aligned} \frac{\partial \phi}{\partial t} + \alpha_1 \nabla^2 \phi &= 0 \text{ on } \Omega \\ \frac{\partial \phi}{\partial x} &= 0 \text{ on sides} \\ \frac{\partial \phi}{\partial y} &= 0 \text{ on bottom} \\ \phi(x, y, T) &= 0 \text{ on } \Omega \end{aligned}$$

We use the Green's Function for heat and the method of images.

Green's Function for Heat

The Green's Function for heat, or the heat kernel, in 2 Dimensions is:

$$K_{(x_0,y_0)}(x,y,t) = \frac{1}{4\pi\alpha t} e^{-\frac{(x-x_0)^2+(y-y_0)^2}{4\alpha t}}$$

It can be shown that this function solves:

$$\begin{aligned} \frac{\partial K_{(x_0,y_0)}}{\partial t} - \alpha\nabla^2 K_{(x_0,y_0)} &= 0 \\ K_{(x_0,y_0)}(x,y,0) &= \delta(x-x_0, y-y_0) \end{aligned}$$

where

$$\delta(a,b) = \begin{cases} 1 & a = b = 0 \\ 0 & \text{otherwise} \end{cases}$$

Test Functions

By carrying out the computations, it can be shown that

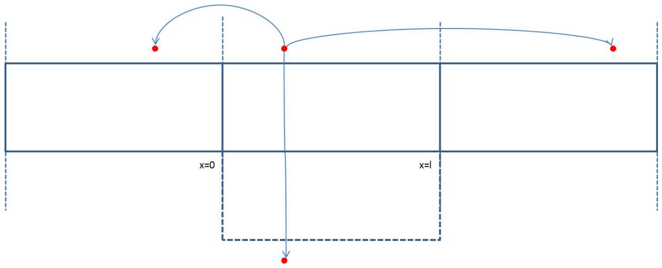
$$\int_0^t K_{(x_0,y_0)}(x, y, \tau)\ d\tau$$

also solves the heat equation, and has zero initial condition.

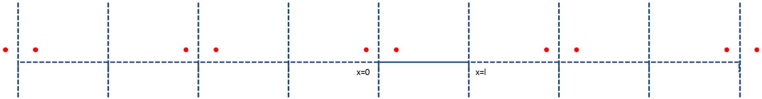
Therefore any sum of these integrals will also solve the heat equation with zero initial condition.

Method of Images

We need to create a function ϕ with zero Neumann Data at $x = 0$, $x = l$, and $y = 0$. To do this, we apply the method of images, making the function (nearly) symmetric about these lines in order to guarantee zero normal derivative there. We choose any point (x_0, y_0) that is off of the top edge of the plate, and use that for our source point.



Method of Images

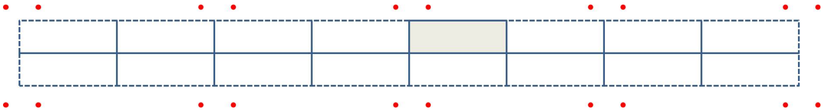


To guarantee zero flux at $x = 0$ we take the point, reflect it over $x = 0$ and then symmetrically reflect it over the lines $x = jl$ for $j \in \{-N, ..., N\}$ for some N. Note that this set of data is almost symmetric around $x = l$, and the only difference is a set of two points more than Nl units away, so the flux at $x = l$ is nearly zero. The equation describing the temperature for this situation is:

$$\sum_{j=-N}^{N} \int_0^t K_{(2jl+x_0,y_0)}(x,y,\tau) + K_{(2kl-x_0,y_0)}(x,y,\tau)\ d\tau$$

Method of Images

Finally, to get the solution with zero Neumann Data at $y = 0$, we reflect all of the points over the line $y = 0$.



Which gives a final equation of

$$\Psi(x,y,t) = \sum_{j=-N}^{N} \int_0^t K_{(2jl+x_0,y_0)}(x,y,\tau) + K_{(2kl-x_0,y_0)}(x,y,\tau) + K_{(2jl+x_0,-y_0)}(x,y,\tau) + K_{(2kl-x_0,-y_0)}(x,y,\tau)\ d\tau$$

Test Functions

However, this is not exactly the function we want. We need something that solves

$$\frac{\partial \phi}{\partial t} + \alpha \nabla^2 \phi = 0 \quad \text{with} \quad \phi(x, y, T) = 0$$

Since Ψ solves the forward heat equation with zero initial condition, defining

$$\phi(x, y, t) = \Psi(x, y, T - t)$$

gives a function with all the desired properties.

Choosing a set of points $(x_0, y_0)_k$ gives the set of functions ϕ_k used in the calculations.