Navier-Stokes Equations: An Introduction

Matt Charnley

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Abstract

The Navier-Stokes equations are some of the most studied partial differential equations because they are important to both theoretical and applied mathematics. Theoretical mathematicians are attempting to prove that these equations admit a unique solution for given sets of initial data, and applied mathematicians use them to model the flow of an incompressible fluid in a variety of situations. But where do these equations come from? In this talk, some of the history of the Navier-Stokes equations and a derivation of them from physical principles will be presented. Then, a few simple problems will be discussed to show how by making some assumptions (which may or may not be accurate), explicit solutions of these equations can be obtained. Finally, modern results will be shown to explain what mathematicians have so far in terms of proving the desired existence and uniqueness results.

1 Introduction

In mathematical contexts, the Navier-Stokes equations are written in the form

$$\frac{\partial u_i}{\partial t} + \sum_{j=1}^{n} u_j \frac{\partial u_i}{\partial x_j} = \nu \Delta u_i - \frac{\partial P}{\partial x_i} + f_i(x,t) \tag{1}$$

$$\text{div } u = 0$$

where the f_i are components of an externally applied force, u_i is the velocity, P is the pressure, and ν is a positive constant dependent on the fluid. The Millennium Problem asks for a proof of existence and uniqueness of solutions in $\mathbb{R}^3$ to these (vectorial) equations that are either smooth and periodic, or smooth with bounded energy. This problem is still unsolved, so we do not know if solutions actually exist or are unique in arbitrary domains/all of Euclidean space.

However, engineers (specifically chemical) have classes devoted to the use of the Navier-Stokes equations to analyze flows through different geometries and under different circumstances. How is this possible? The main thing that allows this is that most engineering problems can be designed/assumed to fit more strict conditions, which then allow terms to be removed from the equation, granting explicit solutions.

The main goal of this talk is to give an introduction to these equations and where they come from. I'll start by introducing the equations and the people that contributed to their development. I will then go into a derivation of the equations and show how they are derived from the conservation of momentum. Next, I'll talk about some of the assumptions that engineers make when using these equations to solve problems. Finally, I'll move back to the mathematics side, where I will discuss some of the more recent results towards this Millennium Problem of proving the existence and uniqueness of solutions.

2 History

Many famous names, and some of our favorite people, were involved in the development of the Navier-Stokes equations. First of all is Newton. Newton's second law, $F = ma$ is a statement of conservation of momentum, which is exactly what gives rise to the Navier-Stokes equations. Euler also got involved by writing down some equations for fluid flow, most of which will be derived from first principles. His equations, however, ignored the viscosity (ν) and external force terms. Cauchy developed the idea of the stress tensor that bears his name.

The Cauchy Stress tensor $\underline{\sigma}$ is a 3×3 matrix defined as follows

$$\sigma_{ij} = \text{Force of fluid of greater } i \text{ on lesser } i \text{ in } j \text{ direction}$$

As an example, take this cubic piece of fluid. σ_{xx} in this case represents the force... This value is negative in compression. The other values are defined the same way σ_{xy} etc. Now, if we take the average compression force over this piece of fluid, we get... the pressure. So, if we define the pressure P as

$$P = -\frac{1}{3}\text{trace}(\underline{\sigma}),$$

we can decompose $\underline{\sigma}$ into

$$\underline{\sigma} = -P\underline{I} + \underline{\tau} \tag{2}$$

where $\underline{\tau}$ is a anisotropic, symmetric, traceless matrix called the deviatoric stress. Cauchy's main work shows that σ, and hence τ are symmetric matrices. His other contribution was to note that the external force exerted on a point of fluid can be represented by

$$\vec{f} = \underline{\sigma} \cdot \vec{n}$$

where $\vec{n}$ is an outward normal to the surface. Newton had already been thinking about these two matrices, but the description of τ was finally flushed out by Stokes to give Newton's Law of Viscosity, which says that, for a Newtonian, incompressible fluid,

$$\tau_{ij} = \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right). \tag{3}$$

where μ is the viscosity of the fluid.

So, what do these two assumptions mean? The fact that the fluid is Newtonian and incompressible is what is assumed in the statement of the Navier-Stokes Equations. The first one, incompressibility, basically means that the density of the fluid is a constant. Even if there is more pressure on one area of the fluid, it does not 'compress' and the density does not increase. This is a very reasonable assumption for liquids. Assuming that the fluid is Newtonian is a little more complicated. Basically, it means that there is a linear relationship between the rate of strain and stress applied to the system. Think of a rubber band. If you pull on it a little, it stretches a little. If you pull about twice as hard, it will stretch twice as far. That's the idea here. For a fluid, it's the rate of deformation that matters as opposed to the amount, but it works the same. The force τ is proportional to the rate of change of velocity, and the constant of proportionality is μ, the fluid viscosity.

With all of this, we can finally introduce the people with their names on these equations. Claude Louis Marie Henri Navier (1785-1836) was a French engineer and physicist. He spent most of his time working at the Corps of Bridges and Roads, and designed several bridges for the Department of the Seine. He also had Fourier as his doctoral advisor and took over for Cauchy as a professor at the Ecole Polytechnique. His main works involved putting the theory of elasticity into a mathematically usable form (1821) and modeling the modulus of elasticity of materials (1826). His work on the equations of Fluid Flow came out in 1822.

Sir George Gabriel Stokes, 1st Baronet (1819-1903) was born in Ireland and spent most of his time working as a professor at Cambridge. Outside of his Fluid Dynamics work, he represented Cambridge in Parliament, was a member of the Royal Society, and helped to bring fame to the mathematical physics department at Cambridge, along with Maxwell and Kelvin. He published work on the polarization of light, rainbows, flourescence, and has the Stokes' theorem from differential geometry, as well as a ton of other things. The unit for kinematic viscosity is called a stoke in his honor. His work on Fluid Dynamics and these equations came out in 1845.

3 Derivation

So, now we're going to see how, using these various developments, we can derive the Navier-Stokes equations from basic principles. This is probably not the way that it was originally derived, but this way is (most likely) easier and, to me at least, makes sense. Let D be an arbitrary domain in $\mathbb{R}^3$. We're going to use vectorial equations on D to derive the Navier-Stokes Equations. We will do this by the use of conserved quantities.

3.1 Mass Balance

Firstly, we know that mass is conserved within our domain. The general formula for a balance is

$$acc = in - out + gen - cons$$

- Mass can not be created or destroyed, so those two terms are zero.
- The accumulation of mass in our domain is given by the expression

$$acc = \frac{\partial}{\partial t}\int_D \rho \, dV = \int_D \frac{\partial \rho}{\partial t} \, dV. \tag{4}$$

- The amount of mass coming in to our out of the domain is determined by the flux over the surface:

$$in - out = -\int_{\partial D} \rho(\vec{u}\cdot\vec{n}) \, dA. \tag{5}$$

However, applying the divergence theorem to (5) gives

$$in - out = -\int_D \nabla \cdot (\rho \vec{u}) \; dV \tag{6}$$

Combining (4) and (6) gives us an equation for the mass balance, which holds over EVERY domain D

$$\int_D \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) \right] \; dV = 0 \tag{7}$$

Now, since this holds over every domain D, it also holds over all subsets of D. Since this integral is always zero, it implies that the function itself is zero. Therefore, differentiating the gradient, we get

$$\frac{\partial \rho}{\partial t} + \vec{u} \cdot \nabla(\rho) + \rho \nabla \cdot \vec{u} = 0 \tag{8}$$

And now, we make our first assumption about the fluid. By assuming the fluid is incompressible (which is true for most liquids) the density becomes a constant independent of position, velocity, and time. Therefore, all of the derivatives of density are zero, and we are left with

$$\nabla \cdot \vec{u} = 0 \tag{9}$$

which is equivalent to a mass balance in an incompressible fluid.

3.2 Momentum Balance

Now, we need to balance the momentum coming in and out of the domain. In addition to the flow terms, we also have a 'generation' term that comes from any outside forces acting on the body, since

$$F = ma = \Delta p.$$

Note that we will be using momentum per unit volume, integrated over the volume of the domain, which is $\rho \vec{u}$. We are filling in the terms of the equation

$$acc = in - out + gen - cons$$

and we'll start by analyzing the momentum in each direction individually.

- The accumulation term is

$$acc = \frac{\partial}{\partial t} \int_D \rho u_x \; dV = \int_D \frac{\partial \rho u_x}{\partial t} \; dV \tag{10}$$

- The convection term is

$$in - out = -\int_{\partial D} (\rho u_x)(\vec{u} \cdot \vec{n}) \; dA \tag{11}$$

  Applying the Divergence Theorem to this gives

$$in - out = -\int_D \nabla \cdot (\rho u_x \vec{u}) \; dV \tag{12}$$

- Finally, we have to deal with the external forces. These consist of two parts, body forces (gravity) and external forces.

$$gen(grav) = \int_D \rho g_x \; dV \tag{13}$$

$$gen(surf) = \int_{\partial D} \underline{\sigma} \cdot \vec{n} \cdot \hat{e}_x \; dA = \int_D \nabla \cdot \underline{\sigma} \cdot \hat{e}_x \; dV \tag{14}$$

By applying the Cauchy Stress Equation, we can simplify this last equation to

$$gen(surf) = \int_D \nabla \cdot (P\underline{I} + \underline{\tau}) \cdot \hat{e}_x \; dV = \int_D \nabla \cdot \left[\begin{pmatrix} P \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} \tau_{xx} \\ \tau_{yx} \\ \tau_{zx} \end{pmatrix} \right] \; dV \tag{15}$$

Evaluating the derivatives gives

$$gen(surf) = \int_D \frac{\partial P}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} \; dV \tag{16}$$

However, using (3) for the definition of τ, this reduces to

$$\begin{aligned} gen(surf) &= \int_D \frac{\partial P}{\partial x} + \mu \left[\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial}{\partial x}\frac{\partial u_x}{\partial x} + \frac{\partial^2 u_x}{\partial y^2} + \frac{\partial^2 u_y}{\partial x \partial y} + \frac{\partial^2 u_x}{\partial z^2} + \frac{\partial^2 u_z}{\partial x \partial z} \right] \; dV & (17) \\ &= \int_D \frac{\partial P}{\partial x} + \mu \left[\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2} + \frac{\partial^2 u_x}{\partial z^2} \right] + \mu \frac{\partial}{\partial x} \left[\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} \right] \; dV & (18) \\ &= \int_D \frac{\partial P}{\partial x} + \mu \Delta u_x + \mu \frac{\partial}{\partial x} \nabla \cdot \vec{u} \; dV & (19) \end{aligned}$$

and the last term in this expression is zero by the mass balance for an incompressible fluid. Therefore, putting all of these terms together, we have

$$\int_D \frac{\partial \rho u_x}{\partial t} dV = -\int_D \nabla \cdot (\rho u_x \vec{u}) \; dV + \int_D \frac{\partial P}{\partial x} + \mu \Delta u_x \; dV + \int_D \rho g_x \; dV \tag{20}$$

Again, since this holds for any domain D, we can drop the integrals to get the microscopic equation

$$\frac{\partial \rho u_x}{\partial t} + \nabla \cdot (\rho u_x \vec{u}) = \frac{\partial P}{\partial x} + \mu \Delta u_x + \rho g_x \tag{21}$$

If we again invoke the fact that the fluid is incompressible, then ρ is a constant, which can be removed from all derivatives, and we can simplify

$$\nabla \cdot (u_x \vec{u}) = u_x (\nabla \cdot \vec{u}) + \vec{u} \cdot (\nabla u_x) = \vec{u} \cdot (\nabla u_x)$$

to give the final equation

$$\rho \left[\frac{\partial u_x}{\partial t} + \vec{u} \cdot (\nabla u_x) \right] = \frac{\partial P}{\partial x} + \mu \Delta u_x + \rho g_x \tag{22}$$

and the same equation holds if you replace x by either y or z. Comparing this to (1) yields that the forces f_i are replaced by gravity, and up to moving ρ to the other side of the equation $\nu = \mu/\rho$. Many different types of forces are possible based on the particular fluid and situation, but just having gravitational force is the most basic type.

4 Problems

So we have these differential equations, and we don't know whether or not they have a unique solution. What can we do with them? Well, the engineers have an answer, and that is to make simplifying assumptions.

- Steady State Flow
- Unidirectional or 2D flow
- Inviscid

The steady state assumption is that the fluid has reached an equilibrium with the applied forces and boundary conditions. It, in general, is not a bad assumption to make for solving for the equilibrium point, but might not actually describe how the fluid is behaving. Some of these assumptions are ok, and others are completely wrong, which we will see as we go along. But, using these assumptions, we can actually solve some problems.

- Plane Couette Flow (p.116 of DTL)
  Assume we have a stationary plate, a bunch of incompressible Newtonian fluid, and a movable plate on top of it that we pull at a velocity V. We will assume that the flow is unidirectional in the x direction (probably ok) and that the system is infinite in the z direction, so all z derivatives are zero. Finally, we assume that it is a steady flow, so all time derivatives are zero. Then, since u_y and u_z are zero, the mass balance equation tells us that

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} = \frac{\partial u_x}{\partial x} = 0 \tag{23}$$

  Now, we can us the three Navier-Stokes equations to actually solve for the velocity profile.

$$\rho\left[\frac{\partial u_y}{\partial t} + u_x\frac{\partial u_y}{\partial x} + u_y\frac{\partial u_y}{\partial y} + u_z\frac{\partial u_y}{\partial z}\right] = \frac{\partial P}{\partial y} + \mu\Delta u_y + \rho g_y \tag{24}$$

$$0+0+0+0 = \frac{\partial P}{\partial y} + 0 + \rho g \tag{25}$$

$$\rho\left[\frac{\partial u_z}{\partial t} + u_x\frac{\partial u_z}{\partial x} + u_y\frac{\partial u_z}{\partial y} + u_z\frac{\partial u_z}{\partial z}\right] = \frac{\partial P}{\partial z} + \mu\Delta u_z + \rho g_z \tag{26}$$

$$0+0+0+0 = \frac{\partial P}{\partial z} + 0 + 0 \tag{27}$$

$$\rho\left[\frac{\partial u_x}{\partial t} + u_x\frac{\partial u_x}{\partial x} + u_y\frac{\partial u_x}{\partial y} + u_z\frac{\partial u_x}{\partial z}\right] = \frac{\partial P}{\partial x} + \mu\Delta u_x + \rho g_x \tag{28}$$

$$0+0+0+0 = \frac{\partial P}{\partial x} + \mu\frac{\partial^2 u_x}{\partial y^2} + 0 \tag{29}$$

$$\frac{\partial^2 u_x}{\partial y^2} = 0 \tag{30}$$

where we have assumed that there is no applied pressure gradient in the x direction. We can then integrate this last equation twice with the boundary conditions

$$u_x\,|_{y=0} = 0 \qquad u_x\,|_{y=h} = V \tag{31}$$

to see that

$$u_x = Ay + B = \frac{V}{h}y \tag{32}$$

so the profile is linear, as expected. These boundary conditions come from the continuum approximation and the "no-slip" condition: The fluid on the boundary of a system has to be moving at the same velocity as the boundary.

- Flow Down an inclined plane
  Assume we have an inclined plane of angle θ, and our fluid is flowing down the plane under the force of gravity. If it is a viscous fluid, it will quickly reach a steady profile of thickness δ (this might be a decent assumption). To start to analyze this problem, we pick a coordinate system that aligns with the slope of the ramp. Again, assuming unidirectional steady flow, with no changes in the z-direction, we have

$$\rho\left[\frac{\partial u_y}{\partial t} + u_x\frac{\partial u_y}{\partial x} + u_y\frac{\partial u_y}{\partial y} + u_z\frac{\partial u_y}{\partial z}\right] = \frac{\partial P}{\partial y} + \mu\Delta u_y + \rho g_y \tag{33}$$

$$0 + 0 + 0 + 0 = \frac{\partial P}{\partial y} + 0 - \rho g\sin(\theta) \tag{34}$$

$$\rho\left[\frac{\partial u_z}{\partial t} + u_x\frac{\partial u_z}{\partial x} + u_y\frac{\partial u_z}{\partial y} + u_z\frac{\partial u_z}{\partial z}\right] = \frac{\partial P}{\partial z} + \mu\Delta u_z + \rho g_z \tag{35}$$

$$0 + 0 + 0 + 0 = \frac{\partial P}{\partial z} + 0 + 0 \tag{36}$$

$$\rho\left[\frac{\partial u_x}{\partial t} + u_x\frac{\partial u_x}{\partial x} + u_y\frac{\partial u_x}{\partial y} + u_z\frac{\partial u_x}{\partial z}\right] = \frac{\partial P}{\partial x} + \mu\Delta u_x + \rho g_x \tag{37}$$

$$0 + 0 + 0 + 0 = 0 + \mu\frac{\partial^2 u_x}{\partial y^2} - \rho g\cos(\theta) \tag{38}$$

$$\frac{\partial^2 u_x}{\partial y^2} = \rho g\cos(\theta) \tag{39}$$

The boundary conditions on that last equation come from the geometry of the situation

$$u_x|_{y=0} = 0 \qquad \tau_{xy}|_{y=\delta} = \mu\frac{\partial u_x}{\partial y}|_{y=\delta} = 0 \tag{40}$$

- Flow in a pipe. In this case, we have flow through a pipe, where the flow is driven by a pressure gradient in the axial direction. We will use cylindrical coordinates to describe the system. Assuming unidirectional flow again (which is not very good in this case), we can look at the u_z equation.

$$\nabla \cdot u = 0 \qquad \Rightarrow \qquad \frac{\partial u_z}{\partial z} = 0 \tag{41}$$

$$\rho\left[\frac{\partial u_z}{\partial t} + u_r\frac{\partial u_z}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_z}{\partial \theta} + u_z\frac{\partial u_z}{\partial z}\right] = \frac{\partial P}{\partial z} + \mu\Delta u_z + \rho g_z \tag{42}$$

$$0 + 0 + 0 + 0 = \frac{\partial P}{\partial z} + \mu\Delta u_z + 0 \tag{43}$$

If we assume that $\frac{\partial P}{\partial z} = -\frac{\Delta P}{L}$ (good assumption), we can expand the Laplacian

$$\Delta u_z = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u_z}{\partial \theta^2} + \frac{\partial^2 u_z}{\partial z^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right) \tag{44}$$

So we are left to solve

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right) = \frac{\Delta P}{\mu L} \tag{45}$$

with boundary conditions

$$u_z|_{r=R} = 0 \qquad u_z|_{r=0} < \infty \tag{46}$$

where the last condition arises because there is a logarithmic term in the solution.

5 Mathematical Results

There are several different ways that people have approached trying to prove the existence an uniqueness of solutions to the Navier-Stokes Equations. Most of the information presented here comes from the work by Caffarelli, Kohn, and Nirenberg.

A general method in trying to solve this for an arbitrary PDE is to prove the existence of weak solutions, and then show that they satisfy some sort of regularity. The first result, an old result from Leray and Hopf, shows that weak solutions to this boundary/initial value problem do exist. However, the main issue with these solutions is the regularity, we still do not know if the velocity u can develop singularities over time, even if the initial and boundary conditions are C^∞.

Another issue with the problem uniqueness. We can try to find solutions in spaces where we know the solution will be unique however, although these solutions exist with some basic assumptions on u_0, they are only valid for a short time interval. These solutions blow up in finite time, so they do not provide a method to study the problem for all time. Therefore, we are forced to look at weak solutions, or solutions that do not satisfy the PDE in the classical sense.

We now want to find a way to analyze "how bad" the solution is, and where things go wrong. To do this, we define a **singular point** if the function $u(x,t)$ is not in L^∞_{loc} in any neighborhood of

(x,t). Wherever u is essentially bounded, we say that (x,t) is a **regular point**. The main results by both Scheffer and Caffarelli, Kohn, and Nirenberg involve the "size" of the set of singular points. In order to do this, we need a couple of definitions.

Definition 5.1. The **Hausdorff k-measure** of a set X is defined as

$$\mathcal{H}^k(X) = \lim_{\delta \to 0} \mathcal{H}^k_\delta(X)$$

where we define

$$\mathcal{H}^k_\delta(X) = \inf \left\{ \sum_{i=1}^{\infty} r_i^k \mid X \subseteq \cup_{i=1}^{\infty} C_i,\ r_i = \mathrm{diam}(C_i) < \delta \right\}.$$

Note: The Hausdorff Dimension of a set is the unique real number k so that $0 < \mathcal{H}^k(S) < \infty$. We also define parabolic cylinders that will be used in defining a new measure on sets.

$$Q_r(x,t) = \{(y,\tau) \mid |y-x| < r,\ t - r^2 < \tau < t\}$$

Definition 5.2. The **Parabolic Hausdorff k-measure** of a set X is defined analogously

$$\mathcal{P}^k(X) = \lim_{\delta \to 0} \mathcal{P}^k_\delta(X)$$

where we define

$$\mathcal{P}^k_\delta(X) = \inf \left\{ \sum_{i=1}^{\infty} r_i^k \mid X \subseteq \cup_{i=1}^{\infty} Q_{r_i},\ r_i < \delta \right\}.$$

With this, we can state the major results that we have for the Navier Stokes Equations. The first is due to Scheffer (1977)

Theorem 5.1. *For $f = 0$, there exists a solution of the Navier-Stokes Equations such that the singular set S satisfies:*

(a) $\mathcal{H}^{5/3}(S) < \infty$

(b) $\mathcal{H}^1(S \cap (\Omega \times t)) < \infty$ *uniformly in t.*

Caffarelli-Kohn-Nirenberg proved their own version of this theorem.

Theorem 5.2. *For a suitable weak solution of the Navier-Stokes equation, the singular set S satisfies $\mathcal{P}^1(S) = 0$.*

In order to use this theorem, they also prove that under certain conditions on Ω, u_0 and f, the Navier-Stokes equations have a suitable solution.

Theorem 5.3. *Assume that $\Omega = \mathbb{R}^3$ or Ω is bounded with smooth boundary, and let $D = \Omega \times (0,T)$. Suppose that for some $q > \frac{5}{2}$,*

$$f \in L^2(D) \cap L^q_{loc}(D) \qquad \nabla \cdot f = 0$$

and that

$$u_0 \in L^2(\Omega) \quad \nabla \cdot u_0 = 0 \quad u_0 \cdot \nu|_{\partial\Omega} = 0.$$

If Ω is bounded, we also require that $u_0 \in W^{2/5}_{5/4}(\Omega)$. Then the Navier-Stokes equations have a weak solution on D whose singular set S satisfies $\mathcal{P}^1(S) = 0$.

Theorem 5.4. *Suppose that $u_0 \in L^2(\mathbb{R}^3)$, $\nabla \cdot u_0 = 0$ and*

$$\int_{\mathbb{R}^3} |u_0|^2 |x| \; dx < \infty$$

Then there exists a weak solution to the N-S equations for $f = 0$ that is regular within the region

$$\{(x,t) \mid |x|^2 t > K_1\}$$

where K_1 depends on the value of the integral and the $2-$norm of u_0.

And finally, we have a brand new ‘result’. Prof. Mukhtarbay Otelbayev, from Kazakhstan has recently claimed to have solved the entire existence and uniqueness problem for the Navier-Stokes equations. He approaches the periodic boundary version of the problem. The paper, however, is in Russian, and until now, there has not been much of a response to it. Most of the math community is waiting for the paper to be translated in order to fully analyze it.