Inverse Problems for the Heat Equation

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Abstract

Inverse problems is an area of partial differential equations that lends itself well to applied problems. In general, these problems involve trying to reconstruct information about a system from a partial solution to a PDE in the system. These types of solution methods could be useful in fields such as structural analysis, where we would like to find information about the inside of a structure without taking it apart. In this talk, I will discuss a specific inverse problem for the heat equation and show how we attempted to solve this problem numerically. This was a joint work with Andrew Rzeznik (MIT) and Dr. Kurt Bryan (Rose-Hulman Institute of Technology) during a Summer 2012 REU.

1 Introduction

Many engineering problems have their roots in a sort of inverse problems. For partial differential equations, one generally knows all of the characteristics about a domain, and is trying to analyze how, for instance, temperature in the domain changes over time. This has some use in engineering problems, in order to determine what material to make a structure out of, but once the structure is in use, the inverse problem has more significance. In this case, we can measure temperature on the surface, but may not know exactly what the thermal properties look like on the inside. Therefore, we need to solve the inverse problem to understand the situation.

In this talk, I will briefly describe what inverse problems are and the different issues that arise in trying to solve them. I will then sketch and motivate the problem we were interested in solving and show the computations that derive the desired relations. I will then discuss the ill-posedness of this problem and show some of our results and conclusions. The work done on this project was a joint work with Andrew Rzeznik (MIT) and Dr. Kurt Bryan (Rose-Hulman) at the Rose-Hulman REU in Summer 2012. It can also be found in the paper "Thermal Detection of Inaccessible Corrosion" in SIAM Undergraduate Research Online, Volume 6.

2 Inverse Problems

In a sense, inverse problems are the opposite of normal problems that one would try to solve in PDE. PDE problems involve generating a (potentially time-dependent) solution to a differential equation given the system parameters. On the other hand, Inverse Problems want to take a (partial) solution and derive some information about the system parameters from it. For example, one could try to determine the thermal diffusivity of a body by applying a series of heat fluxes to it, given the shape of the body. In general, this would seem to be a fairly simple problem; if there was an analytic solution or full numerical data, one could just take derivatives and plug in for the constants that one is trying to solve for. In physical situations however, the problems become more interesting, because one has a very limited amount of data to work with, as you will see with this problem.

There are three main issues to consider with inverse problems: Existence, Uniqueness, and Well-posedness.

(a) For the applied problems we are trying to solve, existence is usually trivial.

(b) Uniqueness is very important, but it has not been proven for the specific problem we were trying to solve. For the problem that I will sketch out in a second, it seems like there should be some sort of uniqueness, possibly given some constraints on the system.

(c) Well-posedness is the key issue for this problem. This is the fact that a small change in the input data leads to a small change in the output. In general, one can try to prove that a given inverse problem method is well-posed, but in our case, we can see numerically that our reconstructed method is ill-posed. Therefore, we require a regularization method to correct this problem.

3 Sketch of Problem

For our particular problem, which deals with corrosion in a metal plate, we will assume that the metal plate is a finite rectangle Ω of length L and height 1, as shown in Figure 1. The rectangle Ω is set in the Cartesian plane $\mathbb{R}^2$ so that $x = 0$ marks the left side of the rectangle, $x = L$ is the right edge, $y = 0$ denotes the bottom of the sample and $y = 1$ indicates the top.

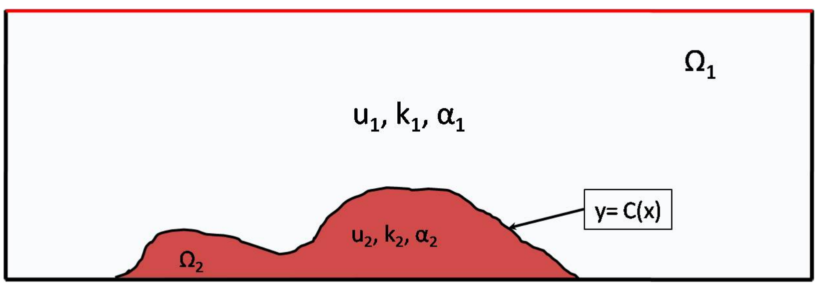

Figure 1: General setup for the problem

We also assume there are two regions, separated by the curve $C(x)$, each having different thermal properties, which include thermal conductivity (k) and thermal diffusivity (α). We will also assume that all external boundaries except the boundary at $y = 1$ are perfectly insulated, so no heat can enter or escape, while some defined heat flux $g(x)$ is applied on the top boundary, $y = 1$. Temperature and heat flux are assumed to be continuous over any interface, including the curve $C(x)$. All of these conditions can be formally stated as follows.

We assume that u_1 satisfies

$$\begin{aligned} \frac{\partial u_1}{\partial t} - \alpha_1 \nabla^2 u_1 &= 0 \text{ on } \Omega_1 \\ \frac{\partial u_1}{\partial x} &= 0 \text{ on } x = 0,\ x = L \\ \frac{\partial u_1}{\partial y} &= g(x) \text{ on } y = 1 \\ u_1(x, y, 0) &= 0 \text{ on } \Omega_1, \end{aligned}$$

while it is assumed that u_2 satisfies

$$\begin{aligned}
\frac{\partial u_2}{\partial t} - \alpha_2 \nabla^2 u_2 &= 0 \text{ on } \Omega_2 \\
\frac{\partial u_2}{\partial x} &= 0 \text{ on } x = 0,\ x = L \\
\frac{\partial u_2}{\partial y} &= 0 \text{ on } y = 0 \\
u_2(x, y, 0) &= 0 \text{ on } \Omega_2,
\end{aligned}$$

and the continuity conditions on $C(x)$ give us

$$\begin{aligned}
u_1 &= u_2 \text{ on } C(x) \\
k_1 \frac{\partial u_1}{\partial \vec{n}} &= k_2 \frac{\partial u_2}{\partial \vec{n}} \text{ on } C(x).
\end{aligned}$$

The forward problem would be stated: Given the input flux $g(x)$, the thermal properties of both materials, and the curve $C(x)$ dividing Ω_1 and Ω_2, find the temperature profiles u_1 and u_2 that satisfy these equations. The inverse problem, on the other hand, assumes that we do not know the curve $C(x)$, but we can measure the temperature on the top surface $u(x, 1, t)$. The idea of this problem is to use this temperature data on the top surface to recover information about the curve $C(x)$.

4 Computation

This will be done using Green's identity and integrating u by parts against strategically chosen test functions. We begin with

Theorem 4.1 (Green's Second Identity)**.** *For any bounded region $D \subset \mathbb{R}^2$ with piecewise smooth boundary ∂D, and any two functions $u, v \in C^2(\bar{D})$, we have*

$$\int_D (u\nabla^2 v - v\nabla^2 u)\ dA = \int_{\partial D} \left(u\frac{\partial v}{\partial \vec{n}} - v\frac{\partial u}{\partial \vec{n}} \right)\ ds.$$

We want to use Green's Identity with what we know about both the interior and the boundary of Ω to generate an approximation for the function $C(x)$. This analysis will also use a collection of 'test functions' ϕ_k, $1 \le k \le M$. Each test function ϕ_k will satisfy

$$\begin{aligned}
\frac{\partial \phi_k}{\partial t} + \alpha_1 \nabla^2 \phi_k &= 0 \text{ on } \Omega, \\
\frac{\partial \phi_k}{\partial \vec{n}} &= 0 \text{ on } y = 0,\ x = 0, \text{ and } x = L, \\
\phi_k(x, y, T) &= 0 \text{ on } \Omega.
\end{aligned}$$

These functions are constructed via the Method of Images, which I will go into later if we have time. We start with the equation

$$\int_0^T \int_{\Omega_1} u_1 \left(\frac{\partial \phi}{\partial t} + \alpha_1 \nabla^2 \phi \right) dA\ dt = 0$$

or

$$\int_0^T \int_{\Omega_1} u_1 \frac{\partial \phi}{\partial t}\ dA\ dt + \alpha_1 \int_0^T \int_{\Omega_1} u_1 \nabla^2 \phi\ dA\ dt = 0.$$

Integrating the first term by parts in time and using Green's Identity on the second term gives

$$\int_{\Omega_1}\left[u_1\phi|_0^T - \int_0^T \phi\frac{\partial u_1}{\partial t}\;dt\right]\;dA + \alpha_1\int_0^T\int_{\Omega_1}\phi\nabla^2 u_1\;dA + \int_{\partial\Omega_1}\left(u_1\frac{\partial\phi}{\partial\vec{n}} - \phi\frac{\partial u_1}{\partial\vec{n}}\right)\;ds\;dt \;=\; 0$$

or

$$\int_{\Omega_1} u_1\phi|_0^T\;dA - \int_{\Omega_1}\int_0^T \phi\left(\frac{\partial u_1}{\partial t} - \alpha_1\nabla^2 u_1\right)\;dt\;dA + \alpha_1\int_0^T\int_{\partial\Omega_1}\left(u_1\frac{\partial\phi}{\partial\vec{n}} - \phi\frac{\partial u_1}{\partial\vec{n}}\right)\;ds\;dt \;=\; 0.$$

The first two terms in the above expression are zero because u_1 solves the heat equation and vanishes at $t = 0$, while ϕ vanishes at $t = T$. Canceling the α_1 gives:

$$\int_0^T\int_{\partial\Omega_1}\left(u_1\frac{\partial\phi}{\partial\vec{n}} - \phi\frac{\partial u_1}{\partial\vec{n}}\right)\;ds\;dt = 0$$

or, factoring in the boundary of Ω_1 and the conditions on the functions there,

$$\int_0^T\int_{top}\left(u_1\frac{\partial\phi}{\partial\vec{n}} - \phi\frac{\partial u_1}{\partial\vec{n}}\right)\;ds\;dt + \int_0^T\int_{C(x)}\left(u_1\frac{\partial\phi}{\partial\vec{n}} - \phi\frac{\partial u_1}{\partial\vec{n}}\right)\;ds\;dt = 0 \tag{1}$$

where $C(x)$ is defined with the downward normal.

Similarly, we can look at the region Ω_2 and start with

$$\int_0^T\int_{\Omega_2} u_2\frac{\partial\phi}{\partial t}\;dA\;dt + \alpha_1\int_0^T\int_{\Omega_2} u_2\nabla^2\phi\;dA\;dt \;=\; 0.$$

Integrating the first term by parts in time and using Green's Identity on the second term gives

$$\int_{\Omega_2}\left[u_2\phi|_0^T - \int_0^T \phi\frac{\partial u_2}{\partial t}\;dt\right]\;dA + \alpha_1\int_0^T\left[\int_{\Omega_2}\phi\nabla^2 u_2\;dA + \int_{\partial\Omega_2}\left(u_2\frac{\partial\phi}{\partial\vec{n}} - \phi\frac{\partial u_2}{\partial\vec{n}}\right)\;ds\right]\;dt = 0.$$

The first term is zero because ϕ and u_2 vanish at the endpoints in time. Since u_2 solves the heat equation in Ω_2, we know that

$$\frac{\partial u_2}{\partial t} = \alpha_2\nabla^2 u_2.$$

Plugging this in above gives

$$\begin{aligned}
-\alpha_2\int_{\Omega_2}\int_0^T\phi\nabla^2 u_2\;dt\;dA + \alpha_1\int_0^T\int_{\Omega_2}\phi\nabla^2 u_2\;dA\;dt + \alpha_1\int_0^T\int_{\partial\Omega_2}\left(u_2\frac{\partial\phi}{\partial\vec{n}} - \phi\frac{\partial u_2}{\partial\vec{n}}\right)\;ds\;dt &= 0\\
(\alpha_1 - \alpha_2)\int_0^T\int_{\Omega_2}\phi\nabla^2 u_2\;dA\;dt + \alpha_1\int_0^T\int_{\partial\Omega_2}\left(u_2\frac{\partial\phi}{\partial\vec{n}} - \phi\frac{\partial u_2}{\partial\vec{n}}\right)\;ds\;dt &= 0\\
\frac{\alpha_1 - \alpha_2}{\alpha_1}\int_0^T\int_{\Omega_2}\phi\nabla^2 u_2\;dA\;dt + \int_0^T\int_{\partial\Omega_2}\left(u_2\frac{\partial\phi}{\partial\vec{n}} - \phi\frac{\partial u_2}{\partial\vec{n}}\right)\;ds\;dt &= 0.
\end{aligned}$$

Taking the boundary of Ω_2 into consideration gives

$$\frac{\alpha_1 - \alpha_2}{\alpha_1}\int_0^T\int_{\Omega_2}\phi\nabla^2 u_2\;dA\;dt + \int_0^T\int_{C(x)}\left(u_2\frac{\partial\phi}{\partial\vec{n}} - \phi\frac{\partial u_2}{\partial\vec{n}}\right)\;ds\;dt + \int_0^T\int_{bottom}\left(u_2\frac{\partial\phi}{\partial\vec{n}} - \phi\frac{\partial u_2}{\partial\vec{n}}\right)\;ds\;dt = 0 \tag{2}$$

where $C(x)$ has the upward normal. $C(x)$ has two different normals in these two equations because Green's Identity treats the boundary of the region as a positively oriented curve with outward normal.

To start, modify equation (1) using the continuity conditions on $C(x)$, to give

$$\int_0^T \int_{top} \left(u_1 \frac{\partial \phi}{\partial \vec{n}} - \phi \frac{\partial u_1}{\partial \vec{n}} \right) \, ds \; dt + \int_0^T \int_{C(x)} \left(u_2 \frac{\partial \phi}{\partial \vec{n}} - \frac{k_2}{k_1} \phi \frac{\partial u_2}{\partial \vec{n}} \right) \, ds \; dt = 0. \tag{3}$$

Then, adding equations (**??**) and (3) gives

$$\int_0^T \int_{top} \left(u_1 \frac{\partial \phi}{\partial \vec{n}} - \phi \frac{\partial u_1}{\partial \vec{n}} \right) \, ds \; dt + \left(1 - \frac{k_2}{k_1} \right) \int_0^T \int_{C(x)} \phi \frac{\partial u_2}{\partial \vec{n}} \, ds \; dt + \frac{\alpha_1 - \alpha_2}{\alpha_1} \int_0^T \int_{\Omega_2} \phi \nabla^2 u_2 \; dA \; dt = 0.$$

Defining

$$RG(\phi) = \int_0^T \int_{top} \left(u_1 \frac{\partial \phi}{\partial \vec{n}} - \phi \frac{\partial u_1}{\partial \vec{n}} \right) \, ds \; dt \tag{4}$$

and rearranging gives

$$RG(\phi) = \left(\frac{k_2}{k_1} - 1 \right) \int_0^T \int_{C(x)} \phi \frac{\partial u_2}{\partial \vec{n}} \, ds \; dt + \frac{\alpha_2 - \alpha_1}{\alpha_1} \int_0^T \int_{\Omega_2} \phi \nabla^2 u_2 \; dA \; dt.$$

Knowing that u_2 solves the heat equation on Ω_2 and that on $C(x)$

$$\vec{n} = \frac{< C'(x), -1 >}{\sqrt{C'(x)^2 + 1}} \qquad \text{and} \qquad ds = \sqrt{C'(x)^2 + 1}$$

this expression can be written as

$$RG(\phi) = \left(\frac{k_2}{k_1} - 1 \right) \int_0^T \int_0^L C'(x) \phi \frac{\partial u_2}{\partial x} \bigg|_{C(x)} - \phi \frac{\partial u_2}{\partial y} \bigg|_{C(x)} dx \; dt + \frac{\alpha_2 - \alpha_1}{\alpha_1 \alpha_2} \int_0^T \int_{\Omega_2} \phi \frac{\partial u_2}{\partial t} \; dA \; dt \tag{5}$$

which is the fully simplified non-linear problem.

We now look to linearize the problem to find a way to numerically solve it. We assume that the corrosion profile, $C(x)$, is small, or

$$C(x) = \epsilon C_0(x)$$

where ϵ is a small positive constant, and $C_0(x)$ is an order 1 function.

The last term in equation (**??**) or (5) may be written as

$$\frac{\alpha_2 - \alpha_1}{\alpha_1 \alpha_2} \int_0^T \int_0^L \int_0^{C(x)} \phi \frac{\partial u_2}{\partial t} \; dy \; dx \; dt.$$

If $C(x)$ is small, we can approximate the function $\phi \frac{\partial u_2}{\partial t}$ by a constant over the innermost integral in y. This term can then be written as approximately equal to

$$\frac{\alpha_2 - \alpha_1}{\alpha_1 \alpha_2} \int_0^T \int_0^L C(x) \; \phi \frac{\partial u_2}{\partial t} \bigg|_{y=0} dx \; dt$$

where we choose to evaluate the functions at $y = 0$.

We also want to linearize the parts of the first integral in (5) about the line $y = 0$, ignoring all terms of $O(\epsilon^2)$. In linearizing these terms, we will use a power series expansion around $y = 0$, namely

$$f|_{y=\gamma} = f|_{y=0} + \gamma \left. \frac{\partial f}{\partial y} \right|_{y=0} + \gamma^2 \left. \frac{\partial^2 f}{\partial y^2} \right|_{y=0} + \cdots,$$

and we will ignore terms of higher orders of ϵ that will vanish because of the linearization assumption. Looking at the terms in (5), we have

$$\begin{aligned}
C'(x)\,\phi\frac{\partial u_2}{\partial x}\bigg|_{C(x)} &= C'(x)\,\phi\frac{\partial u_2}{\partial x}\bigg|_{y=0} + C(x)C'(x)\,\frac{\partial\phi}{\partial y}\frac{\partial u_2}{\partial x}\bigg|_{y=0} + C(x)C'(x)\,\phi\frac{\partial^2 u_2}{\partial x\partial y}\bigg|_{y=0}\\
&= C'(x)\,\phi\frac{\partial u_2}{\partial x}\bigg|_{y=0} + O(\epsilon^2)\\
\phi\frac{\partial u_2}{\partial y}\bigg|_{C(x)} &= \phi\frac{\partial u_2}{\partial y}\bigg|_{y=0} + C(x)\,\frac{\partial\phi}{\partial y}\frac{\partial u_2}{\partial y}\bigg|_{y=0} + C(x)\,\phi\frac{\partial^2 u_2}{\partial y^2}\bigg|_{y=0} + O(\epsilon^2)\\
&= 0 + 0 + C(x)\,\phi\frac{\partial^2 u_2}{\partial y^2}\bigg|_{y=0} + O(\epsilon^2)
\end{aligned}$$

Plugging these terms into equation (5) gives

$$RG(\phi) = \left(\frac{k_2}{k_1}-1\right)\int_0^T\int_0^L C'(x)\,\phi\frac{\partial u_2}{\partial x}\bigg|_{y=0} - C(x)\,\phi\frac{\partial^2 u_2}{\partial y^2}\bigg|_{y=0} dx\ dt + \frac{\alpha_2-\alpha_1}{\alpha_1\alpha_2}\int_0^T\int_0^L C(x)\,\phi\frac{\partial u_2}{\partial t}\bigg|_{y=0} dx\ dt. \tag{6}$$

Integrating the first term in equation (6) by parts in x (integrating $C'(x)$) gives

$$\begin{aligned}
RG(\phi) &= \left(\frac{k_2}{k_1}-1\right)\int_0^T\left[C(x)\phi\frac{\partial u_2}{\partial x}\bigg|_{x=0}^{x=L} + \int_0^L -C(x)\,\frac{\partial\phi}{\partial x}\frac{\partial u_2}{\partial x}\bigg|_{y=0} - C(x)\,\phi\frac{\partial^2 u_2}{\partial x^2}\bigg|_{y=0} - C(x)\,\phi\frac{\partial^2 u_2}{\partial y^2}\bigg|_{y=0} dx\right] dt\\
&\quad + \frac{\alpha_2-\alpha_1}{\alpha_1\alpha_2}\int_0^T\int_0^L C(x)\,\phi\frac{\partial u_2}{\partial t}\bigg|_{y=0} dx\ dt
\end{aligned}$$

where the first term is zero because $\frac{\partial u_2}{\partial x}=0$ at both sides of the rectangle. Combining the two second derivatives into a Laplacian and separating the first term of the integral gives

$$\begin{aligned}
RG(\phi) &= \left(\frac{k_2}{k_1}-1\right)\int_0^T\int_0^L -C(x)\,\frac{\partial\phi}{\partial x}\frac{\partial u_2}{\partial x}\bigg|_{y=0} dx\ dt - \left(\frac{k_2}{k_1}-1\right)\int_0^T\int_0^L C(x)\,\phi\nabla^2 u_2\big|_{y=0}\ dx\ dt\\
&\quad + \frac{\alpha_2-\alpha_1}{\alpha_1\alpha_2}\int_0^T\int_0^L C(x)\,\phi\frac{\partial u_2}{\partial t}\bigg|_{y=0} dx\ dt.
\end{aligned}$$

Converting the second term to a time derivative by the heat equation and reorganizing some terms gives

$$\begin{aligned}
RG(\phi) &= \left(\frac{k_2}{k_1}-1\right)\int_0^T\int_0^L -C(x)\,\frac{\partial\phi}{\partial x}\frac{\partial u_2}{\partial x}\bigg|_{y=0} dx\ dt - \left(\frac{k_2}{k_1}-1\right)\int_0^T\int_0^L C(x)\,\frac{\phi}{\alpha_2}\frac{\partial u_2}{\partial t}\bigg|_{y=0} dx\ dt\\
&\quad + \frac{\alpha_2-\alpha_1}{\alpha_1}\int_0^T\int_0^L C(x)\,\frac{\phi}{\alpha_2}\frac{\partial u_2}{\partial t}\bigg|_{y=0} dx\ dt
\end{aligned}$$

or

$$RG(\phi) = \left(\frac{k_2}{k_1}-1\right)\int_0^T\int_0^L -C(x)\,\frac{\partial\phi}{\partial x}\frac{\partial u_2}{\partial x}\bigg|_{y=0} dx\ dt - \left[\left(\frac{k_2}{k_1}-1\right) - \frac{\alpha_2-\alpha_1}{\alpha_1}\right]\int_0^T\int_0^L C(x)\,\frac{\phi}{\alpha_2}\frac{\partial u_2}{\partial t}\bigg|_{y=0} dx\ dt.$$

Simplifying the coefficient of the second term gives

$$\begin{aligned}
\left(\frac{k_2}{k_1}-1\right) - \frac{\alpha_2-\alpha_1}{\alpha_1} &= \frac{k_2}{k_1} - 1 - \frac{\alpha_2}{\alpha_1} + 1\\
&= \frac{k_2}{k_1} - \frac{\alpha_1}{\alpha_2}
\end{aligned}$$

Thus, our expression becomes

$$RG(\phi) = \left(\frac{k_2}{k_1} - 1\right) \int_0^T \int_0^L -C(x) \left.\frac{\partial \phi}{\partial x} \frac{\partial u_2}{\partial x}\right|_{y=0} dx\ dt - \left(\frac{k_2}{k_1} - \frac{\alpha_2}{\alpha_1}\right) \int_0^T \int_0^L C(x) \left.\frac{\phi}{\alpha_2} \frac{\partial u_2}{\partial t}\right|_{y=0} dx\ dt. \tag{7}$$

Now, making the assumption that the temperature profile u_1 is close to the uncorroded temperature profile u_0, which is reasonable in the case that $C(x)$ is small, we get

$$u_1 = u_0 + \epsilon \tilde{u_1}.$$

We evaluate this expression on the curve $C(x)$ to obtain

$$u_2|_{C(x)} = u_1|_{C(x)} = u_0|_{C(x)} + \epsilon\ \tilde{u_1}|_{C(x)}.$$

Linearizing the far left and far right about $y = 0$ gives

$$u_2|_{y=0} + C(x) \left.\frac{\partial u_2}{\partial y}\right|_{y=0} + O(\epsilon^2) = u_0|_{y=0} + C(x) \left.\frac{\partial u_0}{\partial y}\right|_{y=0} + O(\epsilon^2) + \epsilon\ \tilde{u_1}|_{y=0} + O(\epsilon^2).$$

Since

$$\left.\frac{\partial u_2}{\partial y}\right|_{y=0} = \left.\frac{\partial u_0}{\partial y}\right|_{y=0} = 0$$

we are left with

$$u_2|_{y=0} = u_0|_{y=0} + O(\epsilon).$$

Integrating the last term of equation (7) by parts in time will generate a set of functions which can be used to numerically solve for the function $C(x)$.

$$\begin{aligned}
RG(\phi) &= \left(\frac{k_2}{k_1} - 1\right) \int_0^T \int_0^L -C(x) \left.\frac{\partial \phi}{\partial x} \frac{\partial u_0}{\partial x}\right|_{y=0} dx\ dt - \left(\frac{k_2}{k_1} - \frac{\alpha_2}{\alpha_1}\right) \left[u_0 \phi|_0^T - \int_0^T \int_0^L C(x) \left.\frac{u_0}{\alpha_2} \frac{\partial \phi}{\partial t}\right|_{y=0} dx\ dt \right] \\
&= \left(1 - \frac{k_2}{k_1}\right) \int_0^T \int_0^L C(x) \left.\frac{\partial \phi}{\partial x} \frac{\partial u_0}{\partial x}\right|_{y=0} dx\ dt + \left(\frac{k_2}{k_1} - \frac{\alpha_2}{\alpha_1}\right) \int_0^T \int_0^L C(x) \left.\frac{u_0}{\alpha_2} \frac{\partial \phi}{\partial t}\right|_{y=0} dx\ dt \\
&= \int_0^L C(x) \left[\int_0^T \left(1 - \frac{k_2}{k_1}\right) \left.\frac{\partial \phi}{\partial x} \frac{\partial u_0}{\partial x}\right|_{y=0} + \left(\frac{k_2}{k_1} - \frac{\alpha_2}{\alpha_1}\right) \left.\frac{u_0}{\alpha_2} \frac{\partial \phi}{\partial t}\right|_{y=0} dt \right] dx
\end{aligned}$$

and, defining

$$w_k(x) := \int_0^T \left(1 - \frac{k_2}{k_1}\right) \left.\frac{\partial \phi_k}{\partial x} \frac{\partial u_0}{\partial x}\right|_{y=0} + \left(\frac{k_2}{k_1} - \frac{\alpha_2}{\alpha_1}\right) \left.\frac{u_0}{\alpha_2} \frac{\partial \phi_k}{\partial t}\right|_{y=0} dt$$

we are looking for solutions to the system of integral equations,

$$RG(\phi_k) = \int_0^L C(x) w_k(x) dx \tag{8}$$

with $1 \leq k \leq M$.

4.1 Finding $C(x)$: Least 2-Norm

From the previous section, we are looking for a solution to the system of equations

$$RG_k = \int_0^L C(x) w_k(x)\ dx \tag{9}$$

for $1 \leq k \leq M$.

In order to specify a single solution, we look for the function $C(x)$ with the smallest L^2 norm. The approximation for the function $C(x)$ with this property must be a linear combination of the w_k functions, or

$$C(x) = \sum_{i=1}^{M} \lambda_i w_i(x).$$

Plugging this into equation (9) gives

$$\begin{aligned} RG_k &= \int_0^L \sum_{i=1}^{M} \lambda_i w_i(x) w_k(x)\ dx \\ &= \sum_{i=1}^{M} \lambda_i \int_0^L w_i(x) w_k(x)\ dx. \end{aligned}$$

Defining the coefficient matrix $\mathbf{B}$ by

$$\mathbf{B}_{ik} = \int_0^L w_i(x) w_k(x)\ dx$$

gives

$$RG_k = \sum_{i=1}^{M} \mathbf{B}_{ik} \lambda_i$$

for $1 \leq k \leq M$, or

$$\vec{RG} = \mathbf{B}\vec{\lambda}.$$

Then calculating $\vec{\lambda}$ via $\mathbf{B}^{-1}\vec{RG}$ and letting

$$C(x) = \sum_{i=1}^{M} \lambda_i w_i(x)$$

gives an approximation for the corrosion profile in the system.

5 Ill-posedness

- The matrix B is ill-conditioned, i.e., B has very small eigenvalue, so B^{-1} has very large ones. Therefore, a small variance in the data will give large changes in this value. Also, it makes the function super large.
- Therefore, we need to regularize the matrix in order to get a reasonable result out of this computation. We do this by setting a lower bound on the singular values of B, and removing those from the matrix of B^{-1}.
- Leads to a more accurate reconstruction of the corrosion profile.

6 Results and Conclusions

- Sketch some of the results.
- Averaging over three different fluxes.
- Summarize the overall conclusions.