Graduate Analysis Seminar: Fundamental Solutions

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October 17, 2013

Abstract

Traditionally, separation of variables is the first technique introduced for solving homogeneous partial differential equations. However, this technique is inadequate outside of very simple domains and homogeneous equations. If this method can not be applied, a solution can be developed using fundamental solutions to the differential operator and Greens functions. In this talk, basic distribution theory will be presented in order to define the fundamental solution of a linear differential operator and show how it can be used to solve the Dirichlet problem. The Laplace operator will be discussed in detail to show this construction and what it implies about the regularity of solutions to Laplaces and Poissons Equations.

1 Introduction

If we are given a nice domain and a homogeneous PDE on it, with some boundary/initial conditions, separation of variables can sometimes be used to solve the equation. *Draw box with the heat equation* However, this only works for the most simple situations. For example, if the system is inhomogeneous, separation of variables will be very difficult, if not impossible. The other problem is if the domain is not simple. *Draw crazy domain*. In either of these cases, separation of variables will not be useful in trying to solve the Dirichlet problem. So, we need a new method. In the case of the crazy domain, it may be possible to transform it into a more reasonable region, but another way that works in these regions uses fundamental solutions and Green's functions. Fundamental solutions will let us solve the inhomogeneous problem in all of $\mathbb{R}^n$, and the Green's function will allow us to restrict this solution to a domain and add in a boundary condition.

$$\begin{cases} \Delta u = f & x \in \Omega \\ u = 0 & x \in \partial\Omega \end{cases} \qquad \begin{cases} \Delta u = 0 & x \in \Omega \\ u = g & x \in \partial\Omega \end{cases}$$

Since we want to construct these solutions in complete generality, we need to consider cases where f is not continuous. In these cases, the solution u will not have continuous derivatives of high enough orders to check the solution directly. To deal with this issue, we need the theory of distributions, which will allow us to take the necessary number of derivatives, even if they aren't continuous.

Once we have distributions, we will be able to define the fundamental solution of a differential operator. Finally, from the fundamental solution, we can develop a formulation of Green's functions, which will allow us to solve the Dirichlet problem on a very general level. The analysis here follows that in Folland's book on PDE.

2 Basic Definitions/Distributions

The first step in this process is to develop a small amount of the theory of distributions in order to use them to solve the given differential equations. This idea was originally proposed by Sergei Sobolev in 1935, and was formalized in its entirety by Laurent Schwartz in the 1940s. We begin with some basic definitions. Let Ω be a subset of $\mathbb{R}^n$.

Definition 2.1. A multi-index α is a vector of the form $\alpha = (\alpha_1, ..., \alpha_n)$. $|\alpha| = \sum \alpha_j$ and $\alpha! = \alpha_1! \cdots \alpha_n!$. We denote partial derivatives using multi-indices by

$$\partial^\alpha = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial x_2^{\alpha_2}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}.$$

Definition 2.2. The space of smooth functions in Ω with compact support is denoted $C_c^\infty(\Omega)$. We say that a sequence of function $\phi_j \in C_c^\infty(\Omega)$ converges to ϕ in $C_c^\infty(\Omega)$ if all of the ϕ_j and ϕ are supported in a common compact subset of Ω and $\partial^\alpha \phi_j \to \partial^\alpha \phi$ uniformly for all multi-indices α.

Remark. This definition of convergence in $C_c^\infty(\Omega)$ comes from the locally convex topology on this space.

Definition 2.3. Let u be a linear functional on $C_c^\infty(\Omega)$ (*i.e.* a linear map from $C_c^\infty(\Omega) \to \mathbb{R}$). We denote the value obtained by applying u to $\phi \in C_c^\infty(\Omega)$ by $\langle u, \phi \rangle$.

This is very suggestive notation as we will soon see.

Definition 2.4. A **distribution** is a linear functional on $C_c^\infty(\Omega)$ that is continuous in the sense that if $\phi_j \to \phi$ in $C_c^\infty(\Omega)$, then $\langle u, \phi_j \rangle \to \langle u, \phi \rangle$.

But, what are these distributions? In some instances, they are called "generalized functions" because all reasonable functions are also distributions.

Example 2.1. Any locally integrable function f is a distribution under the action suggested by the notation above

$$\langle f, \phi \rangle = \int_\Omega f \phi \, dx.$$

Since ϕ has compact support, we only need f to be locally integrable for this to be well-defined.

Example 2.2. Any locally finite measure on $\mathbb{R}^n$ is a distribution under the action

$$\langle \mu, \phi \rangle = \int_\Omega \phi \; d\mu.$$

In particular, with this last example, the Dirac delta "function" is a distribution, since it can be defined using the Dirac (point mass) measure at 0. I can write δ and not feel bad about it!

The main goal with considering these distributions was to give us a new class of potential solutions to differential equations. So, we want to apply linear differential operators to them. If T is a continuous linear operator on $C_c^\infty(\Omega)$ in the sense that if $\phi_j \to \phi$, then $T\phi_j \to T\phi$, and there exists another linear operator T' such that for any $\phi, \psi \in C_c^\infty(\Omega)$

$$\int_\Omega (T\phi)\psi \; dx = \int_\Omega \phi(T'\psi) \; dx$$

then we can have T act on distributions by the formula

$$\langle Tu, \phi \rangle = \langle u, T'\phi \rangle.$$

In this case, T' is called the **dual** or **transpose** operator of T. Thus, by considering various operators and how they act on functions in $C_c^\infty(\Omega)$, we can see how to apply them to distributions.

Example 2.3. If T is multiplication by a function f, then $T' = T$, and we can multiply distributions by functions according to the formula

$$\langle uf, \phi \rangle = \langle u, f\phi \rangle.$$

Example 2.4. If $T = \partial^\alpha$, then integration by parts shows that $T' = (-1)^{|\alpha|}\partial^\alpha$. Since the functions all have compact support, all boundary terms vanish.

Example 2.5. Using above. If $T = \Delta$, $T' = \Delta$. Thus we can apply the Laplacian to distributions by the formula

$$\langle \Delta u, \phi \rangle = \langle u, \Delta\phi \rangle.$$

Also, if T is the heat operator, $\partial_t - \Delta$, then $T' = -\partial_t - \Delta$.

Finally, we need to characterize what we mean by a distribution satisfying a linear operator.

Definition 2.5. Two distributions u, v are **equal on** V open set if $\langle u, \phi \rangle = \langle v, \phi \rangle$ for all $\phi \in C_c^\infty(V)$.

Definition 2.6. Given a linear differential operator L, a distribution u is a **weak solution** to L (or solves L in the **weak sense**) if $Lu = 0$ on Ω as distributions.

3 Fundamental Solution

Given a linear differential operator L, we want to develop a way to solve $Lu = f$ in the weak sense on all of $\mathbb{R}^n$.

Definition 3.1. A **Fundamental Solution** to a linear differential operator L is a distribution K such that $LK = \delta$.

This will allow us to solve the inhomogeneous problem, because if $f \in C_c^\infty$, then taking $u = K * f$ gives

$$Lu = L(K * f) = LK * f = \delta * f = f$$

Another property of distributions is that if they have compact support, we can also define a convolution of distributions. Thus, we can similarly solve this problem when f is a distribution with compact support.

One of the most important properties of fundamental solutions is that they exist under certain circumstances. This theorem was proved separately by Bernard Malgrange and Leon Ehrenpreis.

Theorem 3.1 (Malgrange-Ehrenpreis)**.** *Every linear differential operator with constant coefficients has a fundamental solution.*

In order to explore this concept, I will next go on to showing how one would find the fundamental solution for a particular differential operator, the Laplacian. Namely, we are looking for a distribution N that satisfies

$$\langle N, \Delta\phi \rangle = \langle \Delta N, \phi \rangle = \langle \delta, \phi \rangle = \phi(0).$$

Since $\Delta N = \delta$, N needs to be harmonic on $\mathbb{R}^n \setminus \{0\}$. Since the Laplacian commutes with rotations (Fourier Transform commutes with rotations and the symbol of Δ is radial), this distribution should be radial, because the result of $\Delta N = 0$ is also radial. For radial functions, calculations with the Laplacian can be drastically simplified.

Theorem 3.2. *If $f(x) = \phi(r)$, where $r = |x|$, $x \in \mathbb{R}^n$ then*

$$\Delta f(x) = \phi''(r) + \frac{n-1}{r}\phi'(r)$$

Proof. Since $\frac{\partial r}{\partial x_j} = \frac{x_j}{r}$, we can apply the chain rule to get

$$\Delta f(x) = \sum_1^n \frac{\partial}{\partial x_j}\left[\frac{x_j}{r}\phi'(r)\right] = \sum_1^n \left[\frac{x_j^2}{r^2}\phi''(r) + \frac{1}{r}\phi'(r) - \frac{x_j^2}{r^3}\phi'(r)\right] = \phi''(r) + \frac{n}{r}\phi'(r) - \frac{1}{r}\phi'(r).$$

■

Corollary 3.1. *If $f(x) = \phi(r)$ is a radial function, then $\Delta f = 0$ on $\mathbb{R}^n \setminus \{0\}$ if an only if $\phi(r) = a + br^{2-n}$ $(n \neq 2)$ or $\phi(r) = a + b \log r$ $(n = 2)$.*

This statement above suggest a form for the fundamental solution of the Laplacian. Since the constant a is harmonic no matter what, we can ignore it. Therefore, we just need to find the proper constant b such that $\Delta N = \delta$ in the sense of distributions, or the integration gives $\phi(0)$ as opposed to some multiple of this value.

Theorem 3.3. *The fundamental solution to the Laplacian is*

$$N(x) = \frac{|x|^{2-n}}{(2-n)\omega_n} \ (n \neq 2) \quad \text{or} \quad N(x) = \frac{1}{2\pi} \log |x| \ (n = 2)$$

where ω_n is the area of the unit sphere in $\mathbb{R}^n$.

Remark. This function is normally denoted N in honor of Newton, as it is the gravitational potential generated by a unit point mass at the origin.

3.1 Regularity Properties

In particular, this fundamental solution is continuous on $\mathbb{R}^n \setminus \{0\}$, which, by another theorem, implies that Δ is hypoelliptic, which directly gives us that:

Proposition 3.1. *If $f \in C^\infty(\Omega)$, then $u \in C^\infty(\Omega)$.*

There are some other minor conditions on f that will give good properties with respect to solutions of $\Delta u = f$. I will not prove any of these, but will state them here for completeness.

Proposition 3.2. *If $f \in L^1(\mathbb{R}^n)$, then $f * N$ is well defined as a locally integrable function, and $\Delta(f * N) = f$.*

Now, but what about finite orders of differentiability? A first guess would be to say that if $f = C^k(\Omega)$, then $u = C^{k+2}(\Omega)$. However, this fails to be true (in $n > 1$), but by moving to a different set of function spaces, a similar statement does hold.

Proposition 3.3. *Suppose $k \geq 0$, $0 < \alpha < 1$, and Ω is an open set in $\mathbb{R}^n$. If $f \in C^{k+\alpha}(\Omega)$ and u is a distribution solution of $\Delta u = f$ on Ω, then $u \in C^{k+2+\alpha}(\Omega)$.*

So, this gives us a way to solve the inhomogeneous problem on all of $\mathbb{R}^n$. It also shows some of the results of Elliptic Regularity Theory, which says that u will be as regular as f allows it to be.

4 Green's Functions

So, for the last part of the talk, I want to discuss the method restricting these solutions to a given domain, and incorporating the Dirichlet data into the solution. We want to be able to solve the two different types of problems

$$\begin{cases} \Delta w = 0 & x \in \Omega \\ w = g & x \in \partial\Omega \end{cases} \qquad \begin{cases} \Delta v = f & x \in \Omega \\ v = 0 & x \in \partial\Omega \end{cases}.$$

In order to do this, we are going to define the Green's Function for the given domain Ω. In order to simplify notation later on, we write

$$N(x,y) = N(x-y)$$

as the fundamental solution of a linear operator L.

Definition 4.1. The **Green's Function** of the linear differential operator L for the bounded domain Ω with smooth boundary S is the function $G(x,y)$ on $\Omega \times \overline{\Omega}$ with the following properties:

(a) $L(G(x,\cdot) - N(x,\cdot)) = 0$ in Ω and $G(x,\cdot) - N(x,\cdot)$ is continuous in $\overline{\Omega}$.

(b) $G(x,y) = 0$ for each $x \in \Omega$, $y \in S$.

Remark. If G exists, it is unique because for each $x \in \Omega$, it is the solution to the Dirichlet Problem, $LG(x,\cdot) = 0$, $G(x,y) = -N(x,y), y \in S$.

This theory will work for any linear differential operator, but in order to move further into the analysis, we will go back to dealing with the Laplace operator. If we assume that Green's functions exist (which they do, and I will not be proving it here), it is C^∞ on $\Omega \setminus \{x\}$ and we can show that

Lemma 4.1. $G(x,y) = G(y,x)\ \forall\, x,y \in \Omega$.

Proof. For any x,y set $u(z) = G(x,z)$ and $v(z) = G(y,z)$. Then $\Delta u(z) = \delta(x-z)$ and $\Delta v(z) = \delta(y-z)$. Then, formally applying Green's Identity gives

$$\begin{aligned} G(x,y) - G(y,x) &= \int_\Omega G(x,z)\delta(y-z) - G(y,z)\delta(x-z)\ dz \\ &= \int_\Omega G(x,z)\Delta v(z) - G(y,z)\Delta u(z)\ dz \\ &= \int_S G(x,z)\partial_{n_z} G(y,z) - G(y,z)\partial_{n_z} G(x,z)\ d\sigma(z) = 0 \end{aligned}$$

Because the Green's function is zero on the boundary. ∎

Since it is symmetric, we can extend the Green's Function to a function on $\overline{\Omega} \times \overline{\Omega}$ that is also harmonic in the second argument. So, now we can use this Green's Function to solve our problems.

Proposition 4.1. *The functions v and w below satisfy the Dirichlet problems above:*

$$v(x) = \int_\Omega G(x,y)g(y)\ dy \qquad w(x) = \int_S h(y)\partial_{n_y} G(x,y)\ d\sigma(y) = \int_S h(y)\nabla_y G(x,y)\cdot \vec{n}\ d\sigma(y).$$

Proof. The claim for v can be easily proven.

$$\begin{aligned} v(x) &= \int_\Omega G(x,y)f(y)\ dy \\ &= \int_\Omega N(x,y)f(y)\ dy + \int_\Omega [G(x,y) - N(x,y)]f(y)\ dy \\ &= f * N(x) + \int_\Omega [G(x,y) - N(x,y)]f(y)\ dy \end{aligned}$$

and this second part is harmonic in x. Furthermore, for x on the boundary, $v(x) = 0$ because $G(x,y) = 0$. ■

For w, the claim is a little more tricky, because it needs to be shown that w extends continuously to the boundary and is equal to h there.

The function $\partial_{n_y} G(x,y)$ on $\Omega \times S$ is called the **Poisson Kernel** for Ω. This formula is therefore integrating the boundary values of w against this kernel to determine the values everywhere inside Ω.

5 Examples

Here, I will show the example of the Poisson Kernel and Green's function in the unit ball in $\mathbb{R}^n$.

Now, we want to take a unit ball, $B_1(0)$, in $\mathbb{R}^n$, and solve the Dirichlet problem via the Green's function and Poisson kernel. The idea is the same as in the half space, since a unit charge at x can be canceled by a charge at $x/|x|^2$, which is what you get when you reflect x over the sphere. This second charge, however, must have magnitude $|x|^{2-n}$ in order to cancel the first one on the sphere.

Lemma 5.1 ([1] 2.46)**.** *If $x,\ y \in \mathbb{R}^n$, $x \neq 0$, and $|y| = 1$, then*

$$|x - y| = \left| |x|^{-1}x - |x|y \right|.$$

Proof. We have

$$\begin{aligned} |x-y|^2 &= |x|^2 - 2x\cdot y + 1 \\ &= \big||x|y\big|^2 - 2(|x|^{-1}x)\cdot(|x|y) + \big||x|^{-1}x\big|^2 \\ &= \big||x|y - |x|^{-1}x\big|^2. \quad \blacksquare \end{aligned}$$

For the case $n > 2$, we define

$$\begin{aligned} G(x,y) &= N(x,y) - |x|^{2-n}N(|x|^{-2}x, y) \\ &= \frac{1}{(2-n)\omega_n}\left[|x-y|^{2-n} - \big||x|^{-1}x - |x|y\big|^{2-n}\right]. \end{aligned}$$

In looking at the first equation, we can see that $G(x,y) - N(x,y)$ is harmonic if

$$\Delta N(|x|^{-2}x, y) = 0 \quad \Leftrightarrow \quad y \neq |x|^{-2}x$$

However, $||x|^{-2}x| = |x|^{-1}$, so if $x \in B_1(0)$, $|x| < 1$ and $|x|^{-1} > 1$. If $y = |x|^{-2}x$ and $x \in B_1(0)$, then $|y| > 1$, so $y \notin B_1(0)$. Therefore, $G(x,y) - N(x,y)$ is harmonic on $B_1(0)$. Also, if $|y| = 1$, the lemma gives us that $G(x,y) = 0$ by the second equation. A direct calculation can also show that $G(x,y) = G(y,x)$.

In the case $n = 2$ we have a similar formula

$$\begin{aligned} G(x,y) &= \frac{1}{2\pi}\left[\log|x-y| - \log\big||x|^{-1}x - |x|y\big|\right] \quad (x \neq 0) \\ G(0,y) &= \frac{1}{2\pi}\log|y|. \end{aligned}$$

From the Green's function, we can compute the Poisson kernel for solving the Dirichlet problem:

$$P(x,y) = \partial_{\nu_y}G(x,y) \quad x \in B_1(0),\ y \in S_1(0)$$

which can be calculated for $n \geq 2$ using $\partial_{\nu_y} = y\cdot\nabla_y$ on $S_1(0)$ as

$$P(x,y) = \frac{-1}{\omega_n}\left[\frac{y\cdot(x-y)}{|x-y|^n} - \frac{|x|y\cdot(|x|^{-1}x - |x|y)}{||x|^{-1}x - |x|y|^n}\right]$$

Using Lemma 5.1, since $|y| = 1$, this simplifies to

$$P(x,y) = \frac{1-|x|^2}{\omega_n|x-y|^n}.$$

If we want to solve the Dirichlet problem

$$\Delta u = f \text{ on } B_1(0) \qquad u = 0 \text{ on } S_1(0)$$

we can use the function

$$u(x) = \int_{B_1(0)} G(x,y)f(y)\ dy$$

and if we want to solve the dual problem we need the Poisson kernel.

Theorem 5.1 ([1] 2.48)**.** *If $f \in L^1(S_1(0))$ and P is as computed above, set*

$$u(x) = \int_S P(x,y)f(y)\ d\sigma(y) \qquad (x \in B_1(0)).$$

Then u is harmonic on $B_1(0)$. If f is continuous, u extends continuously to $\overline{B_1(0)}$ and $u = f$ on $S_1(0)$. If $f \in L^p(S)(1 \le p < \infty)$, then $u_r \to f$ in the L^p norm as $r \to 1$, where $u_r(y) = u(ry),\ y \in S_1(0)$.

Proof. In this proof, in order to simplify notation, define

$$B := B_1(0) \qquad S := S_1(0).$$

For each $x \in B$, $P(x,y)$ is bounded for all y in S, since $|x-y| \neq 0\ \forall y \in S$, so the function $u(x)$ is well-defined. Since $G(x,y)$, and therefore $P(x,y)$ is harmonic in x, u is also harmonic.

Claim 1 For any $y_0 \in S$ and any neighborhood V of y_0 in S,

$$\lim_{r\to 1} \int_{S\setminus V} P(ry_0, y)\ d\sigma(y) = 0.$$

Proof.

$$P(ry_0, y) = \frac{1-r^2}{\omega_n |ry_0 - y|^n}$$

The denominator of this function is uniformly bounded away from 0 for $|r| < 1$ and outside of a neighborhood of y_0. Thus, the integral is defined, and the integrand goes to 0 as $r \to 1$. ■

Claim 2 $\int_S P(x,y)\ d\sigma(y) = 1\ \forall x \in B$.

Proof. Since P is harmonic in x, we can use the mean value property around 0 to get that

$$\omega_n P(0,y) = \int_S P(ry', y) d\sigma(y')$$

for any $r \in (0,1)$, which defines a sphere around the origin. However,

$$P(0,y) = \frac{1-0}{\omega_n |0-y|^n} = \frac{1}{\omega_n}\ \forall y \in S$$

and the lemma above gives that $P(ry', y) = P(ry, y')$. Setting $x = ry \in B$ gives

$$1 = \omega_n \frac{1}{\omega_n} = \int_S P(x, y')\ d\sigma(y').$$ ■

Suppose that f is continuous on S, which means it is uniformly continuous. Given $\epsilon > 0$, choose $\delta > 0$ such that $|x-y| < \delta \Rightarrow |f(x) - f(y)| < \frac{\epsilon}{2}$. Define $V_x = \{y \in S : |x-y| < \delta \}$. Then, for any $x \in S$, $r < 1$

$$\begin{aligned} |f(x) - u(rx)| &= \left| \int_S [f(x) - f(y)] P(rx, y) \, d\sigma(y) \right| \\ &\leq \sup_{y \in V_x} \{|f(x) - f(y)|\} \int_{V_x} P(rx, y) \, d\sigma(y) + 2 \sup_{x \in S \setminus V_x} \{|f(x)|\} \int_{S \setminus V_x} P(rx, y) \, d\sigma(y) \end{aligned}$$

The first term is less than $\frac{\epsilon}{2}$, since we are on V_x and the integral is less than 1 by claim 2. Similarly, there is an δ' such that if $1 - r < \delta'$, then the value of the integral in the second term is less than $\frac{\epsilon}{4||f||_\infty}$. Therefore, this sum is less than ϵ, and $u_r \to f$ as $r \to 1$, and u extends continuously to $\overline{B}$ with $u = f$ on S.

Finally, suppose $f \in L^p$. Given $\epsilon > 0$, choose $g \in C(S)$ with $||g - f||_p \leq \frac{\epsilon}{3}$, which exists because $C(S)$ is dense in L^p. Setting $v(x) = \int_{S_1(0)} P(x, y) g(y) \, d\sigma(y)$, we have

$$||f - u_r||_p \leq ||f - g||_p + ||g - v_r||_p + ||v_r - u_r||_p.$$

The first term on the right is less than $\frac{\epsilon}{3}$ by definition, and the second term is also less than $\frac{\epsilon}{3}$ if $1 - r$ is small enough. The third term can be shown to be smaller than $\frac{\epsilon}{3}$ by the generalized Young's Inequality, since P is a positive function with integral 1 on S, and $||f - g||_p \leq \frac{\epsilon}{3}$. Therefore, this sum is less than ϵ, and $u_r \to f$ in the L^p norm as $r \to 1$.

■

6 Appendix

Proof 1. Using Green's Identities. Consider the case $n \neq 2$.

Take any $\phi \in C_c^\infty$. Define $\Omega = B_r(0) \setminus B_\epsilon(0)$, removing a small ball around the origin, where r has been chosen such that the support of ϕ is contained in $B_r(0)$.

Then

$$\begin{aligned} \int_\Omega N \Delta \phi \, dx &= \int_\Omega N \Delta \phi - \phi \Delta N \, dx \\ &= \int_{\partial \Omega} N \partial_\nu \phi - \phi \partial_\nu N \, d\sigma \\ &= \int_{S_r(0)} N \partial_\nu \phi - \phi \partial_\nu N \, d\sigma + \int_{S_\epsilon(0)} N \partial_\nu \phi - \phi \partial_\nu N \, d\sigma \end{aligned}$$

where the first line uses the fact that N is harmonic in Ω, and Green's Identity is applied to get to the second line.

Since N is of the form $b|x|^{2-n}$, taking the outward normal of the sphere gives that on $S_\rho(0)$.

$$\begin{aligned} N &= b\rho^{2-n} \\ \partial_\nu N &= b(2-n)\rho^{1-n} \\ \partial_\nu \phi &= \nu \cdot \nabla\phi = \frac{1}{\rho}\sum x_j \partial_j \phi \end{aligned}$$

Plugging this in with $\rho = r$ and $\rho = \epsilon$, gives

$$\int_\Omega N\Delta\phi \; dx = \int_{S_r(0)} br^{2-n}\partial_\nu\phi - \phi\cdot b(2-n)r^{1-n} \; d\sigma + \int_{S_\epsilon(0)} b\epsilon^{2-n}\partial_\nu\phi + \phi\cdot b(2-n)\epsilon^{1-n} \; d\sigma$$

However, since the support of ϕ is contained in $B_r(0)$, both ϕ and $\partial_\nu\phi$ are identically zero on $S_r(0)$. So the first integral vanishes and we are left with

$$\begin{aligned} \int_\Omega N\Delta\phi \; dx &= \int_{S_\epsilon(0)} b\epsilon^{2-n}\frac{1}{\epsilon}\sum_{j=1}^n x_j\partial_j\phi + \phi\cdot b(2-n)\epsilon^{1-n} \; d\sigma \\ &= \frac{b}{\epsilon^{n-1}}\int_{S_\epsilon(0)} \sum_{j=1}^n x_j\partial_j\phi + \phi(2-n) \; d\sigma \\ &= b\omega_n\left[\frac{1}{\epsilon^{n-1}\omega_n}\int_{S_\epsilon(0)}\sum_{j=1}^n x_j\partial_j\phi \; d\sigma + \frac{2-n}{\epsilon^{n-1}\omega_n}\int_{S_\epsilon(0)}\phi \; d\sigma\right] \end{aligned}$$

Then, sending ϵ to zero gives that

$$\begin{aligned} \langle N, \Delta\phi\rangle &= \int_{\mathbb{R}^n} N\Delta\phi = \lim_{\epsilon\to 0}\int_\Omega N\Delta\phi \\ &= \lim_{\epsilon\to 0} b\omega_n\left[\frac{1}{\epsilon^{n-1}\omega_n}\int_{S_\epsilon(0)}\sum_{j=1}^n x_j\partial_j\phi \; d\sigma + \frac{2-n}{\epsilon^{n-1}\omega_n}\int_{S_\epsilon(0)}\phi \; d\sigma\right] \\ &= b\omega_n\left[\lim_{\epsilon\to 0}\frac{1}{\epsilon^{n-1}\omega_n}\int_{S_\epsilon(0)}\sum_{j=1}^n x_j\partial_j\phi \; d\sigma + \lim_{\epsilon\to 0}\frac{2-n}{\epsilon^{n-1}\omega_n}\int_{S_\epsilon(0)}\phi \; d\sigma\right] \\ &= b\omega_n[0 + (2-n)\phi(0)] = b(2-n)\omega_n\phi(0) \end{aligned}$$

where the expression is simplified in the last line because the average value of ϕ on $S_\epsilon(0)$,

$$\frac{1}{\epsilon^{n-1}\omega_n}\int_{S_\epsilon(0)}\phi \; d\sigma \to \phi(0)$$

as $\epsilon \to 0$. Also, since ϕ and $\partial\phi$ are continuous, they are bounded as $\epsilon \to 0$, and $\sum x_j\partial_j\phi$ is of order ϵ as ϵ goes to zero, so that integral vanishes.

Since we want $\langle N, \Delta\phi \rangle = \phi(0)$, we set $b = \frac{1}{(2-n)\omega_n}$, giving

$$N(x) = \frac{|x|^{2-n}}{(2-n)\omega_n}$$

as the fundamental solution. ■

Another proof of the fundamental solution for Δ uses some notation that will be employed later, so it will be shown here.

Proof 2. Smoothed Functions. Consider the case $n \neq 2$.

Define

$$N^\epsilon(x) = \frac{(|x|^2 + \epsilon^2)^{(2-n)/2}}{(2-n)\omega_n}.$$

$N^\epsilon \to N$ pointwise as $\epsilon \to 0$, and N^ϵ are all dominated by a locally integrable function ($|N|$) as $\epsilon \to 0$, so by the dominated convergence theorem $N^\epsilon \to N$ in the topology of distributions. Therefore, we need to show that

$$\Delta N^\epsilon \to \delta \quad \text{ or } \quad \langle \Delta N^\epsilon, \phi \rangle \to \phi(0) \;\; \forall\, \phi \in C_c^\infty \quad \text{ as } \epsilon \to 0$$

Calculation shows that

$$\Delta N^\epsilon(x) = \frac{n}{\omega_n}\epsilon^2(|x|^2 + \epsilon^2)^{-(n+2)/2} = \epsilon^{-n}\psi(\epsilon^{-1}x)$$

where

$$\psi(x) = \Delta N^1(x) = \frac{n}{\omega_n}(|x|^2 + 1)^{-(n+2)/2}.$$

Since the function is radial $\Delta N^\epsilon(-x) = \Delta N^\epsilon(x)$ and

$$\langle \Delta N^\epsilon, \phi \rangle = \int \Delta N^\epsilon(x)\phi(x)\; dx = \int \Delta N^\epsilon(-x)\phi(x)\; dx = \phi * \Delta N^\epsilon(0) \to a\phi(0)$$

where $a = \int \psi(x)\; dx$ by approximations to the identity (Theorem **??**). However, integration in polar coordinates gives

$$\int \psi(x)\; dx = n\int_0^\infty (r^2+1)^{-(n+2)/2} r^{n-1}\; dr = \frac{n}{2}\int_0^1 s^{(n-2)/2}\; ds = 1$$

by the substitution $s = \frac{r^2}{r^2+1}$.

Therefore $\langle \Delta N^\epsilon, \phi \rangle \to \phi(0)$, and N is a fundamental solution of Δ. ■

References

[1] Folland, Gerald B. Introduction to Partial Differential Equations.