Solving First Order PDEs Using Characteristic Strips

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February 18, 2015

Abstract

This talk will discuss a method for solving first order PDEs (in two dimensions) using Integral Surfaces and Characteristic Strips. The general method will be presented in both the non-linear and quasi-linear case, and several examples will be shown. This talk mostly follows chapter 1 of Fritz John.

1 Introduction

The general first order PDE in two dimensions can be written in the form

$$F(x, y, u, u_x, u_y) = 0$$

where F is some arbitrary function, usually with some level of smoothness. Depending on the exact form of the equation, there can be several ways to try to solve it. For instance

$$au_x + bu_y = 0$$

is a form of the transport equation, where u must be constant along lines of the form $bx - ay = c$, since if

$$z(t) = u(at + x_0, bt + y_0)$$

then we have

$$z'(t) = au_x + bu_y = 0.$$

If we are given initial data $u(x, 0) = h(x)$, we then have that

$$u(x, y) = u(at + x, bt + y) = u(a\frac{-y}{b} + x, b\frac{-y}{b} + y) = u(x - \frac{a}{b}y, 0) = h(x - \frac{a}{b}y).$$

Similarly

$$xu_x + yu_y = 0$$

can be solved by switching to polar coordinates, where this equation becomes

$$u_r = 0.$$

However, there is a more general method that works out for all equations of this form. This method has more of a geometric feel, and involves the concept of integral surfaces of the PDE. The idea is, we write $z = u(x, y)$ and interpret this as a two-dimensional surface in $\mathbb{R}^3$. We call this an *integral surface* of the PDE. Then, the PDE will put constraints on what the tangent plane to this surface will look like, giving us a different approach to solving this equation.

This method can be applied to both quasi-linear and fully non-linear PDE, and I will address both over the course of this talk. As we will see, this method reduces solving these PDEs to solving a system of ODEs, where we have standard existence and uniqueness results.

2 Quasi-Linear Equations

2.1 Characteristic Curves

The general quasi-linear equation takes the form

$$a(x, y, u)u_x + b(x, y, u)u_y = c(x, y, u). \tag{1}$$

We call the solution $z = u(x, y)$ the *integral surface* of the PDE. We can also write this in the form

$$a(x, y, z)u_x + b(x, y, z)u_y - c(x, y, z) = 0 = \langle a(x, y, z), b(x, y, z), c(x, y, z)\rangle \cdot \langle u_x, u_y, -1\rangle.$$

These coefficients (a, b, c) define a vector field on (at least a part of) $\mathbb{R}^3$. However, what do we know about this last vector? The tangent plane to the surface $z = u(x, y)$ has the form

$$z - z_0 = u_x(x - x_0) + u_y(y - y_0)$$

which has normal vector $\langle u_x, u_y, -1\rangle$. This tells us that the tangent plane to $z = u(x, y)$ must contain the vector $\langle a, b, c\rangle$. That is, the surface $z = u(x, y)$ must always be tangent to the vector $\langle a, b, c\rangle$ at any point (x, y, z) that belongs to the surface.

In order to build towards this surface, we define the *characteristic curves* of this PDE. The idea is that I can define curves γ in $\mathbb{R}^3$ that have tangent vector $\langle a, b, c\rangle$. Then, if this curve is contained in a surface S, any point along γ clearly satisfies the orthogonality condition, because the tangent plane to S has to contain the tangent vector to γ, which is a multiple of $\langle a, b, c\rangle$.

Definition 2.1. Given a point x_0, y_0, z_0, a *characteristic curve* through $P_0 = (x_0, y_0, z_0)$ is a curve $\gamma(t) = \langle x(t), y(t), z(t)\rangle$ such that $\gamma(0) = P_0$ and

$$x'(t) = a(x, y, z) \qquad y'(t) = b(x, y, z) \qquad z'(t) = c(x, y, z). \tag{2}$$

If a, b, c are C^1, then standard existence and uniqueness theory for ODEs tells us that there is exactly one characteristic curve through any point P_0.

As described before, if a surface S is a union of characteristic curves, then it is an integral surface. We, however, also have the converse. That is, for any integral surface S, S is the union of characteristic curves. This is a consequence of

Theorem 2.1. *Let P_0 lie on an integral surface S, and γ be the characteristic curve through P_0. Then $\gamma \subset S$.*

Proof. Let $\gamma(t) = \langle x(t), y(t), z(t)\rangle$ satisfy $\gamma(t_0) = P_0$. We then form the expression

$$U(t) = z(t) - u(x(t), y(t))$$

Computing, we see that

$$\begin{aligned}
\frac{dU}{dt} &= \frac{dz}{dt} - u_x(x,y)\frac{dx}{dt} - u_y\frac{dy}{dt} \\
&= c(x,y,z) - u_x(x,y)a(x,y,z) - u_y(x,y)b(x,y,z) \\
\frac{dU}{dt} &= c(x,y,U(t)+u(x,y)) - u_x(x,y)a(x,y,U(t)+u(x,y)) - u_y(x,y)b(x,y,U(t)+u(x,y))
\end{aligned}$$

and we also have, by construction $U(t_0) = 0$. Now, $U \equiv 0$ is a particular solution of the above ODE, because $u(x, y)$ is an integral surface. By uniqueness theory for ODEs, this is the only solution that vanishes at t_0. However, $U \equiv 0$ is exactly the statement that the entire characteristic curve lies in S. ■

Consequences of this:

- If two integral surfaces S_1, S_2 intersect at a point P, then they intersect along the entire characteristic curve through P.
- If two integral surfaces intersect along a curve, then that curve is characteristic.

2.2 The Cauchy Problem

With this set-up, we want to look at solving the Cauchy problem for the PDE (1). This problem is specified (for a first order equation) by giving some data along a curve in the xy-plane, i.e. $u(f(s), g(s)) = h(s)$ for some function h. Rewriting this, we see that there must be a curve Γ

$$x = f(s) \qquad y = g(s) \qquad z = h(s)$$

that we want to belong to our particular integral surface. Note, the "initial value problem" has the specific form

$$x = s \qquad y = 0 \qquad z = h(s)$$

and so this is a generalization of that problem.

So, how can we build our integral surface from this curve? Assuming that Γ is not a characteristic curve of the PDE near some point $P_0 = \Gamma(s_0)$, we know that the integral surface must contain the characteristic curves through each point of Γ. So, we build them. For each s near s_0, I can solve the ODEs (2) generating functions $X(s,t)$, $Y(s,t)$, $Z(s,t)$ solving

$$\begin{cases}
\frac{\partial X}{\partial t}(s,t) = a(X(s,t), Y(s,t), Z(s,t)) & X(s,0) = f(s) \\
\frac{\partial Y}{\partial t}(s,t) = b(X(s,t), Y(s,t), Z(s,t)) & Y(s,0) = g(s) \\
\frac{\partial Z}{\partial t}(s,t) = c(X(s,t), Y(s,t), Z(s,t)) & Z(s,0) = h(s)
\end{cases}$$

Again, general theory on existence and uniqueness of solutions to ODEs and continuous dependence on parameters gives that there exist C^1 solutions X, Y, and Z for (s,t) "close enough" to $(s_0, 0)$. Now, if we could solve these equations for x and y, namely, if we could find $s = S(x,y)$ and $t = T(x,y)$, we would then have our solution in the form of

$$u(x,y) = Z(S(x,y), T(x,y).$$

The implicit function theorem gives us this result. We have that there are solutions $S(x,y)$ and $T(x,y)$ to

$$x = X(S(x,y), T(x,y)) \qquad y = Y(S(x,y), T(x,y))$$

[that is,

$$0 = X(s,t) - x \qquad 0 = Y(s,t) - y$$

as maps from $\mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$] satisfying $S(x_0, y_0) = s_0$, $T(x_0, y_0)$ if the Jacobian

$$J = \begin{vmatrix} X_s(s_0,0) & Y_s(s_0,0) \\ X_t(s_0,0) & Y_t(s_0,0) \end{vmatrix} = \begin{vmatrix} f'(s_0) & g'(s_0) \\ a(x_0,y_0,z_0) & b(x_0,y_0,z_0) \end{vmatrix} \neq 0.$$

This condition guarantees that $\Sigma : z = u(x,y)$ is locally a surface. It is clearly an integral surface because $\gamma(t) = (X(\cdot,t), Y(\cdot,t), Z(\cdot,t))$ are characteristic curves, which implies that the tangent plane to Σ must contain the vector a, b, c, and so the surface satisfies the PDE.

If this determinant is zero, then we can generate the relations

$$f'b - g'a = 0 \qquad h' = f'u_x + g'u_y \qquad c = au_x + bu_y$$

Rearranging these, we see that

$$bh' - cg' = 0 \qquad ah' - cf' = 0$$

which implies that (a,b,c) is proportional to (f', g', h'). Thus, the curve Γ must be characteristic at P_0. This is a problem, because if the curve is characteristic locally, then there are infinitely many solutions near P_0.

Example 2.1. (a) Simple example: Transport type equation from earlier.

$$au_x + bu_y = 0$$

with given data $u(x,0) = h(x)$.

We then have the characteristic equations

$$\begin{cases} \frac{\partial X}{\partial t}(s,t) = a & X(s,0) = s \\ \frac{\partial Y}{\partial t}(s,t) = b & Y(s,0) = 0 \\ \frac{\partial Z}{\partial t}(s,t) = 0 & Z(s,0) = h(s) \end{cases}$$

which leads to the solutions

$$X(s,t) = at + s \qquad Y(s,t) = bt \qquad Z(s,t) = h(s)$$

Solving out for t and s, we have that

$$t = y/b \qquad s = x - at = x - \frac{a}{b}y$$

so we have the solution

$$u(x,y) = Z(s,t) = h(x - \frac{a}{b}y)$$

which is the same solution we got earlier.

(b) Non-linear equation

$$u_y - xuu_x = 0$$

with given data $u(x,0) = x$.

$$\begin{cases} \frac{\partial X}{\partial t}(s,t) = -xz & X(s,0) = s \\ \frac{\partial Y}{\partial t}(s,t) = 1 & Y(s,0) = 0 \\ \frac{\partial Z}{\partial t}(s,t) = 0 & Z(s,0) = s \end{cases}$$

which leads to the solutions

$$X(s,t) = se^{zt} \qquad Y(s,t) = t \qquad Z(s,t) = s$$

giving the implicit solution

$$x = ue^{uy}$$

3 Non-Linear Equations

3.1 Monge Cones and Characteristic Curves

From here on out, we use the notation $z = u(x,y)$, $p = u_x$, and $q = u_y$. The most general first order equation has the form

$$F(x,y,z,p,q) = 0$$

The idea here is the same as before, but the dependence on p and q is no longer linear, so we don't have as simple of relations between them. Instead of the characteristic curves from before, we need something called characteristic strips in order to represent the geometry here.

Any integral surface through $P_0 = (x_0, y_0, z_0)$ must have tangent plane

$$z - z_0 = p(x - x_0) + q(y - y_0)$$

for which $F(x_0, y_0, z_0, p, q) = 0$. In the same way as before, this relation restricts the possible tangent planes that $z = u(x,y)$ can have. Before, we had that $\langle a,b,c\rangle \perp \langle p,q,-1\rangle$, but now, we have a 1-parameter family of tangent planes that the integral surface can have at the point P_0. In general, such a one-parameter family can be expected to envelope a cone with vertex at P_0, which is called the *Monge Cone* at P_0. Each possible tangent plane touches the Monge Cone along an edge. Thus, the original PDE defines a *field of cones*, where every integral surface must be tangent to the field of cones at each point.

Example 3.1. (a) The Eikonal equation: $|\nabla u|^2 = 0$, or, in this formulation, $p^2 + q^2 = 1$.

In this case, we have an easy way to represent the 1-parameter family here, setting $p = \cos\theta$, $q = \sin\theta$. Thus, at any point $P_0 = (x_0, y_0, z_0)$, we have that the possible normal vectors to the tangent plane are $\langle \cos\theta, \sin\theta, -1\rangle$. [Sketch a picture] Thus, we see that in this case, we actually do get a cone with vertex at P_0. If the equation was more irregular, then you would not get a circular cone.

(b) The quasi-linear case from before (1). In this case, we have the form

$$ap + bq = c$$

Thus, given p, we can find q via $q = \frac{c-ap}{b}$ assuming b is not zero. Otherwise, we can solve for p given q. The original relation tells us that $\langle p, q, -1\rangle \perp \langle a, b, c\rangle$. Thus, this family of planes is all planes that contain $\langle a, b, c\rangle$. In this case, the Monge Cone degenerates to the vector $\langle a, b, c\rangle$.

From the fact that the Monge Cone is an "envelope" of a family of surfaces, we have that the tangent plane with normal $\langle p, q, -1\rangle$ at the point P_0 satisfies

$$dz = p\ dx + q\ dy \qquad 0 = dx + \frac{dq}{dp}\ dy$$

The first equation comes from the tangent plane. The second comes from the envelope concept, but it can be seen by taking the derivative of the tangent plane equation in p, where we view p as the parameter defining the 1-parameter family, with q defined in terms of p. However, the fact that $F(x_0, y_0, z_0, p, q) = 0$ and this holds for any p, q that satisfy the desired relation, we have that

$$F_p + \frac{dq}{dp} F_q = 0.$$

Combining these relations, we see that

$$dz = p\ dx + q\ dy \qquad \frac{dx}{F_p} = \frac{dy}{F_q}.$$

This defines a direction field on (a subset of) $\mathbb{R}^3$, so long as we already know the surface. We define the characteristic curves belonging to S as those that fit this direction field. By using a parameter t to denote the curve, we have that,

$$\frac{dx}{dt} = F_p \qquad \frac{dy}{dt} = F_q \qquad \frac{dz}{dt} = pF_p + qF_q.$$

In the quasi-linear case, we have that $F_p = a$, $F_q = b$, and $pF_p + qF_q = c$, giving that this reduces to that case when the equation is quasi-linear. Thus, we are fit to call these the characteristic curves of this equation.

As long as we know the surface, these equations are well defined. If we do not know the surface, then this system is still underdetermined; we need two more equations. By differentiating the PDE in x and y, we see that

$$\begin{aligned} F_x + u_x F_z + u_{xx} F_p + u_{xy} F_q &= 0 \\ F_y + u_y F_z + u_{xy} F_p + u_{yy} F_q &= 0 \end{aligned}$$

Then, by the chain rule, we have that

$$\begin{aligned} \frac{dp}{dt} &= u_{xx}\frac{dx}{dt} + u_{xy}\frac{dy}{dt} = u_{xx}F_p + u_{xy}F_q = -F_x - pF_z \\ \frac{dq}{dt} &= u_{xy}\frac{dx}{dt} + u_{yy}\frac{dy}{dt} = u_{xy}F_p + u_{yy}F_q = -F_y - qF_z \end{aligned}$$

This fills out a system of 5 ODEs for the functions (x, y, z, p, q). So we have a fully determined system!

Lemma 3.1. *F is an integral of the system. That is* $\frac{dF}{dt} = 0$.

Proof. This is just a computation

$$\begin{aligned} \frac{dF}{dt} &= F_x\frac{dx}{dt} + F_y\frac{dy}{dt} + F_z\frac{dz}{dt} + F_p\frac{dp}{dt} + F_q\frac{dq}{dt} \\ &= F_xF_p + F_yF_q + F_z(pF_p + qF_q) + F_p(-F_x - qF_z) + F_q(-F_y - qF_z) = 0 \quad \blacksquare \end{aligned}$$

We call the system of five ODEs with the statement $F(x, y, z, p, q) = 0$ the *characteristic equations.*

3.2 Characteristic Strips

A solution to the characteristic equations is a set of five equations $(x(t), y(t), z(t), p(t), q(t))$. We call any quintuple of numbers (x, y, z, p, q) a *plane element*, and we interpret it as a point (x, y, z) and the tangent plane

$$\zeta - z = p(\xi - x) + q(\eta - y)$$

Such an element is called *characteristic* if $F(x, y, z, p, q) = 0$.

A one-parameter family of elements $(x(t), y(t), z(t), p(t), q(t))$ is called a *strip* if the elements are tangent to the curve $(x(t), y(t), z(t))$; that is, if the 'strip condition' is satisfied

$$\frac{dz}{dt} = p\frac{dx}{dt} + q\frac{dy}{dt},$$

that is, if the tangent plane defined by $\langle p, q, -1\rangle$ is also tangent to the curve at $(x.y.z)$. If this family solves the characteristic equations (the five ODEs, and the relation that $F(x, y, z, p, q) = 0$), then it is called a *characteristic strip*.

A surface $z = u(x, y)$ can be thought of as a two parameter family of elements

$$(x(s,t), y(s,t), z(s,t), p(s,t), q(s,t))$$

where, in order for these elements to make up a surface, the 'strip conditions' must be satisfied in both directions

$$\frac{\partial z}{\partial t} = p\frac{\partial x}{\partial t} + q\frac{\partial y}{\partial t} \qquad \frac{\partial z}{\partial s} = p\frac{\partial x}{\partial s} + q\frac{\partial y}{\partial s}$$

To visualize these planes, if they satisfy the strip conditions, they fit together kind of like scales on a fish. Infinitesimally small scales.

Once we define these strips, we have some similar results to the characteristic curves from earlier:

(a) A characteristic strip is determined uniquely from any one of its elements (ODE uniqueness).

(b) Given a point, the strip consists of these characteristic curve through the point and the tangent planes to S along that curve.

(c) If two integral surfaces touch at a point, then it touches along the characteristic curve.

3.3 Cauchy Problem

As before, we view the Cauchy Problem as specifying a curve Γ that must belong to the integral surface. We let Γ be defined by

$$x = f(s) \qquad y = g(s) \qquad z = h(s).$$

We again proceed by passing suitable characteristic strips through Γ, and extending those to a surface to make the solution.

Let $P_0 = \Gamma(s_0)$ and assume that f, g, h are C^1 near s_0. We first need to complete Γ into a characteristic strip by finding functions $\phi(s)$ and $\psi(s)$ so that

$$h'(s) = \phi(s)f'(s) + \psi(s)g'(s) \qquad F(f, g, h, \phi, \psi) = 0$$

Solutions to this equation may not exist, and may not be unique if they exist. In order to move forward, we assume that we are given a p_0, q_0 so that

$$h'(s_0) = p_0 f'(s_0) + q_0 g'(s_0) \qquad F(x_0, y_0, z_0, p_0, q_0) = 0 \qquad \Delta = f'(s_0)F_q - g'(s_0)F_p \neq 0$$

This Δ is the determinant of the matrix needed to apply the inverse function theorem in order to prove that there exist such a ϕ and ψ. Namely, we have the function

$$\tilde{F}(s, \phi, \psi) = \begin{cases} \phi f'(s) + \psi g'(s) - h'(s) \\ F(f(s), g(s), h(s), \phi, \psi) \end{cases}$$

where $\tilde{F}(s_0, p_0, q_0) = 0$ and the Jacobian determinant is exactly Δ.

Therefore, we have a one-parameter family of elements $(x(s), y(s), z(s), \phi(s), \psi(s))$, defined near a parameter value s_0. As with the characteristic curves in the quasi-linear case, we now pass a characteristic strip through each element that reduces to the standard element for $t = 0$. Thus, we solve the ODE system

$$\begin{cases} \frac{\partial X}{\partial t}(s,t) = F_p(X,Y,Z,P,Q) & X(s,0) = f(s) \\ \frac{\partial Y}{\partial t}(s,t) = F_q(X,Y,Z,P,Q) & Y(s,0) = g(s) \\ \frac{\partial Z}{\partial t}(s,t) = PF_p(X,Y,Z,P,Q) + QF_q(X,Y,Z,P,Q) & Z(s,0) = h(s) \\ \frac{\partial P}{\partial t}(s,t) = -F_x(X,Y,Z,P,Q) - PF_z(X,Y,Z,P,Q) & P(s,0) = \phi(s) \\ \frac{\partial Q}{\partial t}(s,t) = -F_y(X,Y,Z,P,Q) - QF_z(X,Y,Z,P,Q) & Q(s,0) = \psi(s) \end{cases}$$

for all $|s-s_0|$ and $|t|$ sufficiently small. Since it is satisfied at $(s_0, 0)$ we have that $F(X,Y,Z,P,Q) = 0$. As before, the functions $X(s,t), Y(s,t), Z(s,t)$ form a parametric representation of this surface. If we can solve for $s = S(x,y)$ and $t = T(x,y)$ as before, we will have an equation (locally) for this surface. We do this again by the implicit function theorem, which works since we have

$$\begin{vmatrix} X_s(s_0,0) & Y_s(s_0,0) \\ X_t(s_0,0) & Y_t(s_0,0) \end{vmatrix} = \begin{vmatrix} f'(s_0) & g'(s_0) \\ F_p & F_q \end{vmatrix} = \Delta \neq 0$$

Thus, we have our equation in the form

$$u(x,y) = z = Z(S(x,y), T(x,y)).$$

Now, if we have an integral surface, we can write the equations in this way. It remains to show that this parametrization gives an integral surface. This will be true if we have $P = u_x$ and $Q = u_y$. We can determine u_x and u_y as parametrized by s and t, via the chain rule, namely

$$Z_s = u_x X_s + u_y Y_s \qquad Z_t = u_x X_t + u_y Y_t$$

Thus, we will be done if we can show that

$$Z_s = PX_s + QY_s \qquad Z_t = PX_t + QY_t$$

which are just the strip conditions for (X,Y,Z,P,Q), that is, this determines if these elements make up a surface. The t equation is just a consequence of the characteristic equations that (X,Y,Z,P,Q) satisfy. For the s equation, we define a new function

$$A(s,t) = Z_s - PX_s - QY_s \qquad A(s,0) = h'(s) - \phi(s)f'(s) - \psi(s)g'(s) = 0$$

for all s. By using the characteristic equations, we see that

$$\begin{aligned} A_t &= Z_{st} - P_t X_s - PX_{st} - Q_t Y_s - QY_{st} \\ &= \frac{\partial}{\partial s}(Z_t - PX_t - QY_t) + P_s X_t - P_t X_s + Q_s Y_t - Q_t Y_s \\ &= 0 + P_s F_p + X_s(F_x + PF_z) + Q_s F_q + Y_s(F_y + QF_z) \\ &= F_x X_s + F_y Y_s + F_p P_s + F_q Q_s + F_z(PX_s + QY_s) \\ &= \frac{\partial F}{\partial s} + F_z(PX_s + QY_s - Z_s) = -F_z A \end{aligned}$$

Thus, by integration, we have that

$$A(s,t) = A(s,0)\ \exp\left(-\int F_z\right) = 0.$$

Thus, we have our desired relation, and (X, Y, Z, P, Q) form an integral surface.

Example 3.2. (a) Take the PDE

$$u_x^2 + u_y^2 = u^2.$$

We want to find the characteristic strips and two specific solutions. For the first part of this, we are going to take arbitrary initial values $(x_0, y_0, z_0, p_0, q_0)$ and solve for the strips. This equation can be rewritten as

$$F(x,y,z,p,q) = p^2 + q^2 - z^2 = 0$$

We then want to solve the strip equations

$$\begin{cases} \frac{\partial X}{\partial t} = F_p(X,Y,Z,P,Q) & X(0) = x_0 \\ \frac{\partial Y}{\partial t} = F_q(X,Y,Z,P,Q) & Y(0) = y_0 \\ \frac{\partial Z}{\partial t} = PF_p(X,Y,Z,P,Q) + QF_q(X,Y,Z,P,Q) & Z(0) = z_0 \\ \frac{\partial P}{\partial (}t) = -F_x(X,Y,Z,P,Q) - PF_z(X,Y,Z,P,Q) & P(0) = p_0 \\ \frac{\partial Q}{\partial t} = -F_y(X,Y,Z,P,Q) - QF_z(X,Y,Z,P,Q) & Q(0) = q_0 \end{cases}$$

Plugging in the derivatives of the function F, we see that we are trying to solve

$$\begin{cases} \frac{\partial X}{\partial t} = 2P & X(0) = x_0 \\ \frac{\partial Y}{\partial t} = 2Q & Y(0) = y_0 \\ \frac{\partial Z}{\partial t} = 2P^2 + 2Q^2 = 2Z^2 & Z(0) = z_0 \\ \frac{\partial P}{\partial (}t) = 2PZ & P(0) = p_0 \\ \frac{\partial Q}{\partial t} = 2QZ & Q(0) = q_0 \end{cases}$$

The Z equation is decoupled, so we can solve it first, to get

$$\frac{dZ}{Z^2} = 2dt \qquad \Rightarrow \qquad -\frac{1}{Z} = 2t + C = 2t - \frac{1}{z_0} \qquad \Rightarrow \qquad Z(t) = \frac{z_0}{1 - 2tz_0}.$$

Now that we have Z, we can solve the P and Q equations, which are identical

$$\frac{dP}{P} = 2Z\ dt = \frac{2z_0\ dt}{1 - 2tz_0} = -\frac{du}{u} \qquad u = 1 - 2tz_0$$

Thus, we have

$$\ln(P) = -\ln(u) + C \qquad P = \frac{C}{u} = \frac{C}{1 - 2tz_0} \qquad \Rightarrow \qquad P(t) = \frac{p_0}{1 - 2tz_0}.$$

And finally, the X and Y equations can be solved.

$$dX = \frac{2p_0}{1-2tz_0}\,dt = -\frac{p_0}{z_0}\frac{du}{u} \qquad \Rightarrow \qquad Z(t) = -\frac{p_0}{z_0}\ln(1-2tz_0) + x_0$$

Thus, we have the system of equations

$$\begin{cases} X(t) = -\frac{p_0}{z_0}\ln(1-2tz_0) + x_0 \\ Y(t) = -\frac{q_0}{z_0}\ln(1-2tz_0) + y_0 \\ Z(t) = \frac{z_0}{1-2tz_0} \\ P(t) = \frac{p_0}{1-2tz_0} \\ Q(t) = \frac{q_0}{1-2tz_0} \end{cases}$$

Now, we want to look at two Cauchy Problems. First we want to solve with the curve

$$\Gamma : x = \cos(s) \qquad y = \sin(s) \qquad z = 1.$$

We need to find values for $\phi(s)$ and $\psi(s)$ in order to solve the equations. By the equation we have that

$$\phi(s)^2 + \psi(s)^2 = 1$$

Using the strip condition in s, we have that

$$\phi(s)(-\sin(s)) + \psi(s)(\cos(s)) = 0$$

We therefore have solutions of the form $\phi(s) = \cos(s)$ and $\psi(s) = \sin(s)$ or both terms can be negative. Then, we have

$$X(s,t) = \frac{\mp\cos(s)}{1}\ln(1-2t) + \cos(s) = (1 \mp \ln(1-2t))\cos(s) \qquad Y(s,t) = (1 \mp \ln(1-2t))\sin(s))$$

We then have

$$X^2 + Y^2 = (1 \mp \ln(1-2t))^2$$

and

$$Z(s,t) = \frac{1}{1-2t} \qquad \Rightarrow \qquad \ln(Z(s,t)) = -\ln(1-2t)$$

Putting these together, we have that

$$X^2 + Y^2 = (1 \pm \ln(Z))^2 \qquad \Rightarrow \qquad Z = \exp\left(\mp(1 - \sqrt{X^2+Y^2}\right)$$

For the second problem, we want to use the curve

$$\Gamma : x = s \qquad y = 0 \qquad z = 1$$

Solving for ϕ and ψ yields the two equations

$$\begin{cases} \phi^2 + \psi^2 = 1 \\ \phi = 0 \end{cases}$$

giving that $\phi = 0$, $\psi = \pm 1$. Thus, the characteristic equations reduce to

$$\begin{cases} X(s,t) = 0 + s \\ Y(s,t) = -\frac{\pm 1}{1} \ln(1-2t) \\ Z(s,t) = \frac{1}{1-2t} \\ P(s,t) = 0 \\ Q(s,t) = \frac{\pm 1}{1-2t} \end{cases}$$

Thus, we have $\ln(Z) = \pm Y$. Thus we have $Z = \exp(\pm Y)$.