Forward and Inverse Problems: An Exploration of Differential Equations

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COS-JAM 2013

May 3, 2013

Outline

PDE's are Everywhere!

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$$\rho\left(\frac{\partial u}{\partial t} + u \cdot \nabla u\right) = \nabla P - \mu \Delta u + \rho g$$

Motivation

Take a domain Ω in $\mathbb{R}^n$ and assume u is a solution of the heat equation on Ω:

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- Existence of solutions

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- Predictive modeling of physical systems
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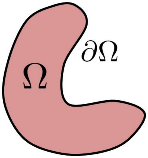
Inverse Problem

- Corrosion detection/Safety analysis
- Deeper understanding of the underlying processes

Forward Problem



We know this region in entirety, and we want to solve

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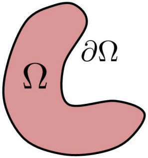

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We will solve this by finding $u = u_1 + u_2$ where

$$\Delta u_1 = f \text{ on } \Omega \qquad u_1 = 0 \text{ on } \partial\Omega$$
$$\Delta u_2 = 0 \text{ on } \Omega \qquad u_2 = g \text{ on } \partial\Omega$$

Green's Function

The main method we will use to solve the forward problem is via Green's Functions.

Definition (Green's Function)

Let $\Omega \subset \mathbb{R}^n$ be a domain, $\partial\Omega$ the boundary, and $\overline{\Omega} = \Omega \cup \partial\Omega$. The **Green's Function for** Δ on Ω is a function

$$G(x,y) : \Omega \times \overline{\Omega} \to \mathbb{R}$$

satisfying

1. $\Delta_y G(x,y) = \delta_y(x) \forall x, y \in \Omega$
2. $G(x,y) = 0$ if $x \in \Omega$, $y \in \partial\Omega$.

Solving the Problem

The Green's Function exists for all Ω smooth boundary (2.35 in Folland).

It can also be proven that the Green's Function is symmetric, *i.e.* $G(x, y) = G(y, x)$ for all x, y. Therefore, G can be extended to a function on $\overline{\Omega} \times \overline{\Omega}$.

So, assuming we have the Green's Function, we can solve the full Forward Problem!

Solving the Problem

To solve

$$\Delta u_1 = f \text{ on } \Omega \qquad u_1 = 0 \text{ on } \partial\Omega$$

Set

$$u_1(x) = \int_\Omega G(x,y)f(y)dy$$

Then, we formally have

$$\begin{aligned}
\Delta_x u_1(x) &= \int_\Omega \Delta_x G(x,y)f(y)dy \\
&= \int_\Omega \delta_x(y)f(y)dy \\
&= f(x) \\
u_1(x) &= \int_\Omega G(x,y)f(y)dy = 0 \text{ if } x \in \partial\Omega
\end{aligned}$$

Solving the Problem

Similarly, if we want to solve

$$\Delta u_2 = 0 \text{ on } \Omega \qquad u_2 = g \text{ on } \partial\Omega$$

we can set

$$u_2(x) = \int_{\partial\Omega} g(y)\partial_{\nu_y} G(x,y) d\sigma(y)$$

where ∂_{ν_y} is the normal derivative on the boundary of Ω and σ denotes integrating over the surface of $\partial\Omega$.

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where ∂_{ν_y} is the normal derivative on the boundary of Ω and σ denotes integrating over the surface of $\partial\Omega$.

However... finding G is hard.

Examples

For simple domains, however, G can be determined explicitly.

For the unit ball $\Omega = \{|x| < 1\}$ in $\mathbb{R}^n$, the Green's Function can be calculated to be

$$G(x,y) = \begin{cases} \frac{1}{(2-n)\omega_n}\left[|x-y|^{2-n} - \left||x|^{-1}x - |x|y\right|^{2-n}\right] & n \neq 2 \\ \frac{1}{2\pi}\left[\log|x-y| - \log\left||x|^{-1}x - |x|y\right|\right] & n = 2 \end{cases}$$

with

$$\partial_{\nu_y} G(x,y) = \frac{1-|x|^2}{\omega_n|x-y|^n}$$

Example 1

Solve

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \text{ on } \Omega \qquad u(x,y) = x \text{ on } \partial\Omega$$

Using the integration formula gives

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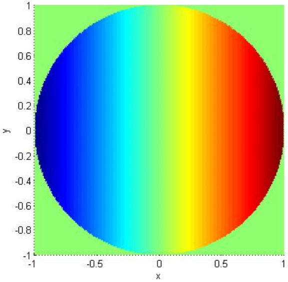

Example 2

Solve

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \text{ on } \Omega \qquad u(x,y) = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases} \quad \text{on } \partial\Omega$$

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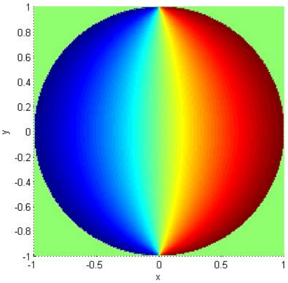

Example 3

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$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = xy \text{ on } \Omega \qquad u(x,y) = \sin(\pi y) \text{ on } \partial\Omega$$

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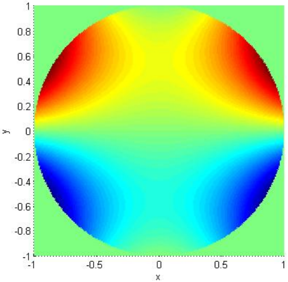


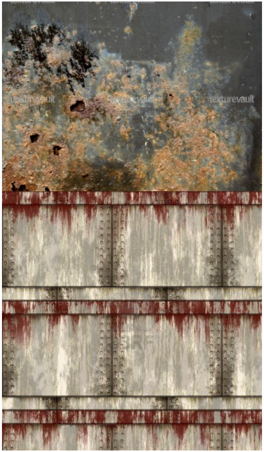
Inverse Problem

As stated before, the Inverse Problem is the case where you want to solve for the properties of the domain from knowing part of the solution.

So, you know the temperature on part of the boundary and want to find out the thermal diffusivity in the domain.

This work was done with Andrew Rzeznik and Dr. Kurt Bryan at the Rose-Hulman 2012 Summer REU.

Our Problem



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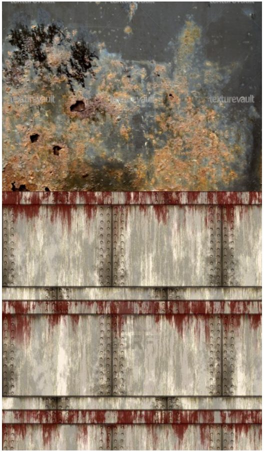

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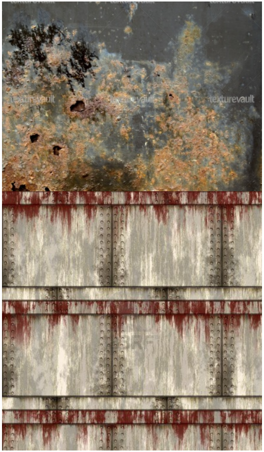

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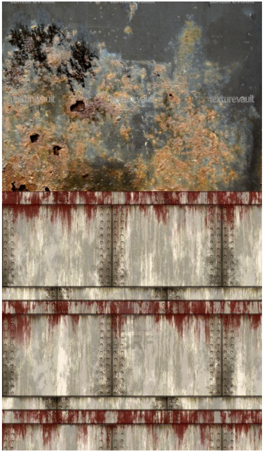


We want to be able to detect corrosion in metal by applying heat and taking temperature measurements on the accessible face. Possible applications include:

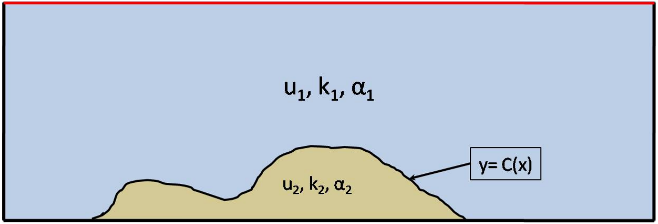
- Detecting external hull corrosion on a vessel from the inside,
- Finding possible corrosion in a chemical pipe, or
- Testing ceramic materials for imperfections.

Our Problem

This problem mathematically reduces to finding how the thermal diffusivity of the metal changes as a function of position.

In our case, we have assumed that we know the thermal diffusivity of both the corroded and uncorroded material, and just need to find the point where it switches.

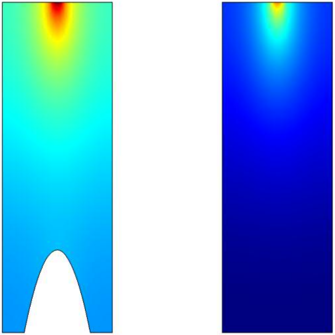
Problem Setup



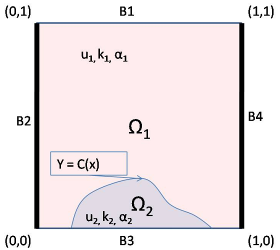
So, we know all of the thermal properties of Ω_1 and Ω_2, and now just need to find the curve $C(x)$ that divides them.

Corrosion Example

Let's take a look at the value of the solution in the corroded vs. uncorroded case. Below is an example of a corrosion profile.



Assumptions



1. There is a defined heat flux at the top of the metal
2. All other sides are insulated (no flux in or out)
3. We can measure the temperature on the top side of the metal
4. The heat equation is satisfied in both regions with their respective thermal quantities

Goal: Find out where the corrosion is on the reverse face of the plate.

Mathematical Approach

We want to use what we know about the interior and boundaries to approximate the curve $C(x)$.

In order to do this, we generate a set of test functions ϕ_k that satisfy

$$\partial_t \phi_k + \alpha \Delta \phi_k = 0$$

And use that to start with

$$\int_0^T \int_\Omega u \left(\partial_t \phi_k + \alpha \Delta \phi_k \right) \, dA \, dt = 0$$

Mathematical Approach

Using integration by parts and using a linearization assumption gives that

$$\int_0^T \int_{\text{top}} \tilde{u}_1 \frac{\partial \phi_k}{\partial \vec{n}} \, ds \, dt$$

$$= \int_0^L C(x) \underbrace{\left[\int_0^T \left(1 - \frac{k_2}{k_1}\right) \frac{\partial \phi_k}{\partial x} \frac{\partial u_0}{\partial x}\bigg|_{y=0} + \left(\frac{k_2}{k_1} - \frac{\alpha_2}{\alpha_1}\right) \frac{u_0}{\alpha_2} \frac{\partial \phi_k}{\partial t}\bigg|_{y=0} dt \right]}_{w_k(x)} dx$$

where the entire left hand side is known from collected temperature data and the determined test function. The w_k functions are also known completely, since u_0 is the uncorroded temperature profile.

Mathematical Approach

With one more assumption, we can approximate $C(x)$ by

$$C(x) = \sum_j \lambda_j w_j(x)$$

and we can solve for λ because

$$a_k = \int_0^T \int_{\text{top}} \tilde{u}_1 \frac{\partial \phi_k}{\partial \vec{n}}\, ds\, dt = \sum_j \lambda_j \int_0^l w_k(x) w_j(x)\, dx = \sum_j \lambda_j B_{jk}$$

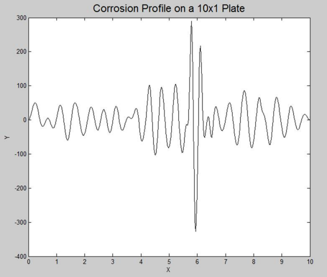
can be recast in a linear algebra problem.

$$\vec{a} = \mathbf{B}\vec{\lambda}$$

Results

So, by taking the inverse of **B**, we can solve for λ and get an approximation to $C(x)$.

However, the problem is ill-posed, so a regularization method is needed to create meaningful solutions.



Results

The coming slides show the major results from our work. For all of the results shown,

$$\begin{aligned} k_1 &= 1 \\ \alpha_1 &= 1 \end{aligned}$$

The thermal parameters for the corroded region will be shown on each graph.

Results: Single Corrosion Profile

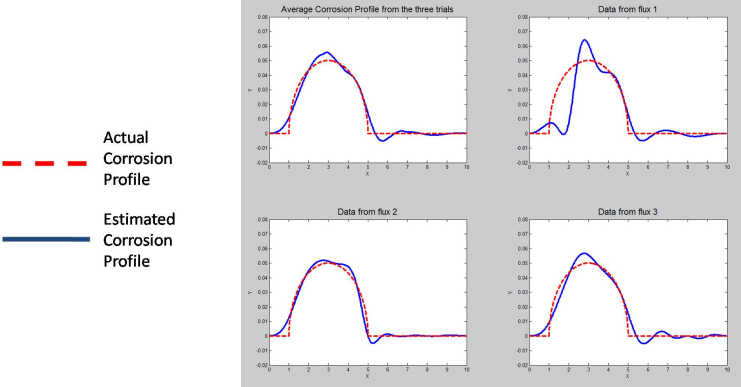
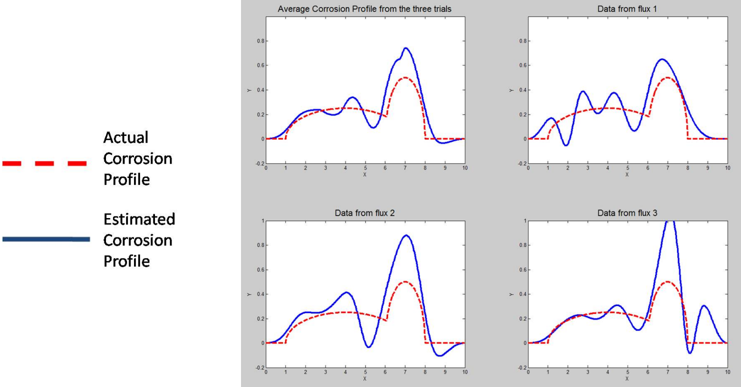


Figure : $k_2 = 0.1$ and $\alpha_2 = 0.1$

Results: Multiple Corrosion Profiles



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Results: Changing Parameters

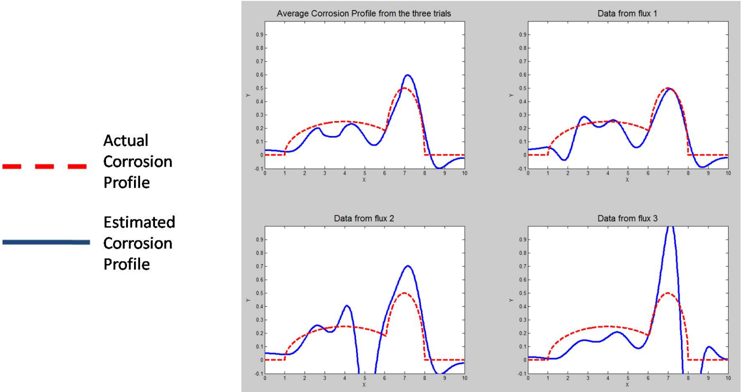
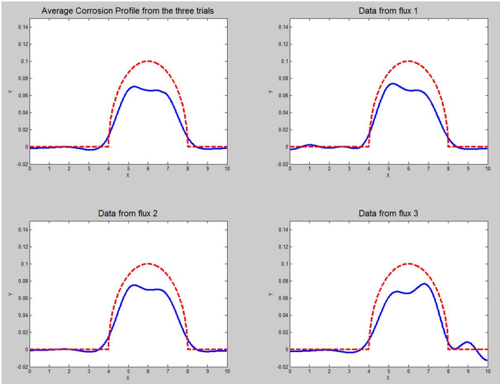


Figure : $k_2 = 0.1$ and $\alpha_2 = 0.05$

Results: Changing Parameters

Actual Corrosion Profile

Estimated Corrosion Profile



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Results: Discussion

- We can find corrosion!!
- We can definitely see where it is, and can estimate the area of the corrosion within an acceptable error.
- Even for large corrosion, which is well outside the range of linearization, we can see where the majority of the corrosion is.
- Reconstruction only takes about a minute after the test functions have been computed.

Future Work

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2. Other ways of utilizing the time dependence of the heat equation.
3. Time-Dependent Fluxes
4. Full 3-Dimensional problem

Questions?

References

[1] Court Hoang and Katherine Osenbach
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