Math 152 - Worksheet 19

Taylor Series

Learning Problems

These problems should be completed on your own. If you need hints on solving a problem, there are some provided with each problem. These are provided on the following pages, with one 'level' of hint per page, with the earlier ones giving away less of the problem than the later ones. Try to work from the earlier hints to the later ones, as this will give you the practice you need to succeed in this class.

1. Find the Maclaurin Series for the function $f(x) = x^2 e^{x^2}$ and determine the interval on which it is valid.
2. Find the Taylor series of $f(x) = e^{3x}$ centered at $c = -1$ and determine the interval on which it is valid.
3. Find the Taylor Series of $f(x) = \dfrac{1}{(x-8)^2}$ centered at $c = 4$ and determine the interval where it is valid.
4. Express the definite integral $\displaystyle\int_0^1 \tan^{-1}(x)\, dx$ as an infinite series. This will be a series of numbers because the answer should be a number.
5. Use Taylor Series to find the value of $f^{(9)}(0)$ for the function $f(x) = xe^{-x^2}$.

Submission Problems

1. Find the Maclaurin Series for the function $f(x) = (x^2 + 1)\cos(2x)$.
2. Find the Taylor Series for $f(x) = \sin x$ centered at $c = \pi$.

Hint #1

1. We can build up this series from one we alread know, namely the series for e^x.

2. We can't use our known series for e^x here, because we aren't centered at zero. So we need to work this out using derivatives.

3. You could work this out directly using derivatives, or you could work with the series and relate it to something we already know about. (I would recommend the latter.)

4. To do this, we need to write $\tan^{-1}(x)$ as a power series, and then integrate it. How do we get to inverse tangent as a power series?

5. These derivatives show up as the coefficients of the Taylor Series, so if we find the Taylor Series for this function, we can get the values of the derivatives at zero.

Hint #2

1. We can plug x^2 into $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ and then multiply by x^2 to get the series we want.

2. Take a few derivatives of $f(x) = e^{3x}$ and see if you can find a pattern for the derivatives.

3. Consider the function $g(x) = \frac{1}{8-x}$. What is the derivative of this function? How can I write this function as a power series centered at 4? To do this you'll want to get it into the form $\frac{1}{1-a(x-4)}$ for some constant a.

4. We can find the power series for $\frac{1}{1+x^2}$ using the tricks from 10.6, and then integrate this to get to inverse tangent.

5. The Taylor Series for $f(x)$ is $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n+1}$.

Hint #3

1. For the interval where it is valid, apply the ratio test to this series. What happens in the limit?

2. The pattern is that $f^{(k)}(x) = 3^k e^{3x}$ so that $f^{(k)}(-1) = 3^k e^{-3}$. Then we can write the series.

3. Since $g(x) = \frac{1}{4}\frac{1}{1-\frac{1}{4}(x-4)}$, we can write $g(x) = \dfrac{1}{4}\displaystyle\sum_{n=0}^{\infty}\frac{1}{4^n}(x-4)^n$.

4. Since $\dfrac{1}{1+x^2} = \displaystyle\sum_{n=0}^{\infty}(-1)^n x^{2n}$, we know that $\tan^{-1}(x) = \sum_{n=0}^{\infty}\frac{(-1)^n}{2n}x^{2n+1}$. We then need to integrate this one more time for our desired series.

5. To get the coefficient of the x^9 term, we need $n = 4$. The coefficient of this term is $\frac{(-1)^4}{4!}$.

Hint #4

1. Since the limit goes to zero, this series converges everywhere. We also know that the series for e^x converges everywhere.

2. Use the ratio test to determine where this series is valid. It should look like the series for e^x

3. Then, we can differentiate this series term by term to get the series for $f(x)$. For the interval where it is valid, we know where the series for $g(x)$ is valid.

4. The series we get is $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n(2n+1)} x^{2n+2}$. Then we can plug in 1 to evaluate this series.

5. The coefficient of this term is $\frac{f^{(9)}(0)}{9!}$, and we can then solve for $f^{(9)}(0)$.

Answers

1. We can write the series as $\sum_{n=0}^{\infty} \frac{1}{n!} x^{2n+2}$ and this is valid for all x.

2. The Taylor Series is $\sum_{n=0}^{\infty} \frac{3^n}{n!} e^{-3}(x+1)^n$ and it converges for all x.

3. The Taylor Series for this f is $\frac{1}{4} \sum_{n=1}^{\infty} \frac{n}{4^n}(x-4)^{n-1}$ and it is valid on $(0,8)$.

4. This integral is $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n(2n+1)}$.

5. $f^{(9)}(0) = \frac{9!}{4!} = 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 = 15120$