Math 152 - Worksheet 20

Taylor Polynomials

Learning Problems

These problems should be completed on your own. If you need hints on solving a problem, there are some provided with each problem. These are provided on the following pages, with one 'level' of hint per page, with the earlier ones giving away less of the problem than the later ones. Try to work from the earlier hints to the later ones, as this will give you the practice you need to succeed in this class.

1. Calculate the Taylor Polynomials $T_3(x)$ and $T_4(x)$ for the function $f(x) = \tan x$ centered at $\frac{\pi}{4}$.
2. Find the Taylor Polynomial $T_4(x)$ centered at $c = -2$ for the function $f(x) = e^x$.
3. Determine the maximum possible error is using $T_2\left(\frac{\pi}{12}\right)$ to approximate $\cos\left(\frac{\pi}{12}\right)$ where the Taylor Polynomial is centered at $c = 0$. Evaluate both $T_2\left(\frac{\pi}{12}\right)$ and $\cos\left(\frac{\pi}{12}\right)$ and confirm that this meets the error bound.
4. How many terms of the Maclaurin Series for $f(x) = \ln(1+x)$ are needed to approximate the value of $\ln(1.2)$ to within 0.0001?
5. Use $T_6(x)$ to approximate $\displaystyle\int_0^1 e^{-x^2}\,dx$.

Submission Problems

1. Find the Taylor Polynomial $T_5(x)$ for the function $\sqrt{x}$ centered at $a = 9$.
2. Use the Error Bound to find a value N for which the error in using $T_n(-0.1)$ to approximate $e^{-0.1}$ to within 10^{-9}, where T_n is the Taylor polynomial centered at 0.

Hint #1

1. To do this, we need the value of the function and it's first four derivatives at $\frac{\pi}{4}$.

2. We need to take derivatives of this function and evaluate at -2 to build this polynomial.

3. To find T_2, we need to either take derivatives of the cosine function, or just use the Taylor Series and cut it off after 2 terms.

4. To do this, we need to find the Maclaurin Series of $\ln(1+x)$. We should do this by computing derivatives.

5. First, we need to find $T_6(x)$ as a function. We can do this using the Taylor Series for e^{-x^2}.

Hint #2

1. For this function $f(x)$, $f'(x) = \sec^2 x$, $f''(x) = 2\sec^2 x \tan x$ and you can continue to compute more derivatives.

2. The derivatives of $f(x)$ are all e^x. Therefore the value of these derivatives is always e^{-2}.

3. We get that $T_2(x) = 1 - \frac{x^2}{2}$.

4. For this $f'(x) = \frac{1}{1+x}$, $f''(x) = \frac{-1}{(1+x)^2}$ and the pattern keeps going.

5. The series for e^{-x^2} is $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n}$ so $T_6(x)$ is this series up to degree 6.

Hint #3

1. Then plugging in $\frac{\pi}{4}$ gives that $f(\frac{\pi}{4}) = 1$, $f'(\frac{\pi}{4}) = 2$, $f''(\frac{\pi}{4}) = 4$ and so on.

2. Then you can write out the polynomial by dividing by $n!$ at each step.

3. Since $f'''(x) = \sin(x)$, we know that this is bounded by 1, so we can use $K = 1$ in our error formula.

4. With the pattern, we get that $f^{(k)}(x) = \frac{(-1)^k (k-1)!}{(1+x)^k}$. Thus, if we want to bound the kth derivative on $[1, 1.1]$, we can bound it by $(k-1)!$

5. So $T_6(x) = 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6}$. We can then integrate this function and plug in 0 and 1.

Hint #4

1. Then, divide each term by $n!$ before writing out the polynomials.

3. The error bound is then $\dfrac{1(\frac{\pi}{12}-0)^3}{3!}$.

4. Thus, for the error bounds, we get that the error in using $T_k(x)$ is $\frac{(k-1)!(0.2)^{k+1}}{(k+1)!}$

5. This will need the reduction formulas and the fact that the antiderivative of $\sec(\theta)$ is $\ln|\sec\theta+\tan\theta|$ to finish up the problem.

Answers

1. $T_3(x) = 1 + 2\left(x - \frac{\pi}{4}\right) + 2\left(x - \frac{\pi}{4}\right)^2 + \frac{8}{3}\left(x - \frac{\pi}{4}\right)^3$ and $T_4(x) = 1 + 2\left(x - \frac{\pi}{4}\right) + 2\left(x - \frac{\pi}{4}\right)^2 + \frac{8}{3}\left(x - \frac{\pi}{4}\right)^3 + \frac{10}{3}\left(x - \frac{\pi}{4}\right)^4$.

2. $T_4(x) = e^{-2}\left(1 + (x-2) + \frac{1}{2}(x-2)^2 + \frac{1}{6}(x-2)^3 + \frac{1}{24}(x-2)^4\right)$

3. The error bound we get is 0.00299. $\cos(\pi/12) = 0.96593$ and $T_2(\pi/12) = .96573$, so we are good.

4. 4 terms

5. $1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} = \frac{26}{35} = 0.7429$. The actual integral is 0.7468.