Math 152 - Worksheet 15

Series with Positive Terms

Learning Problems

These problems should be completed on your own. If you need hints on solving a problem, there are some provided with each problem. These are provided on the following pages, with one 'level' of hint per page, with the earlier ones giving away less of the problem than the later ones. Try to work from the earlier hints to the later ones, as this will give you the practice you need to succeed in this class.

1. Use the integral test to determine if $\sum_{n=4}^{\infty} \frac{n^2}{(n^3+4)^5}$ converges or diverges.

2. Use the Direct Comparison Test to determine if $\sum_{n=3}^{\infty} \frac{n^2+3n+2}{n^3-3}$ converges or diverges.

3. Use the Limit Comparison Test to determine if $\sum_{n=2}^{\infty} \frac{n^4-3n^2+10}{(n^3+3n^2+2n+1)^2}$ converges or diverges.

4. Determine if the series $\sum_{n=1}^{\infty} \frac{(\ln(n))^{25}}{n^2}$ converges or diverges.

5. Determine if the series $\sum_{n=3}^{\infty} \frac{1}{n(\ln(n))^2-n}$ converges or diverges.

6. Determine if the series $\sum_{n=6}^{\infty} \frac{\sin(1/n)}{\sqrt{n}}$ converges or diverges.

Submission Problems

1. Determine if the series $\sum_{n=4}^{\infty} \frac{1}{n\ln n}$ converges or diverges.

2. Determine if the series $\sum_{n=1}^{\infty} \frac{\cos^2 n}{n^2}$ converges or diverges.

Hint #1

1. What integral do you need to compute to determine if this series converges or diverges?

2. Based on the terms on the top and bottom of this fraction, should this series converge or diverge?

3. Think about what the dominant terms in the numerator and denominator are. Use this to decide what b_n should be.

4. A key fact for proving this is the fact that for any positive power a, $n^a > \ln(n)$ for n large enough. (Like the large enough that we need for the Direct Comparison Test)

5. First, think about the series $\sum_{n=3}^{\infty} \frac{1}{n(\ln(n))^2}$, since this should be the dominant term for the series we care about.

6. This is a series with positive terms because as $n \to \infty$, $\sin(1/n) \to 0$ and is positive on this range. What can we do with $\sin(1/n)$ as $n \to \infty$?

Hint #2

1. If we replace the n by x in the expression and integrate, we need to know if $\displaystyle\int_4^\infty \frac{x^2}{(x^3+4)^5}\,dx$ converges or diverges.
2. The gut reaction is that this series should diverge, since n^2 over n^3 should go like $1/n$. How can I make this series smaller and still diverge?
3. Based on this, we should look at $b_n = \dfrac{n^4}{n^6} = \dfrac{1}{n^2}$, which means we will hopefully prove that this series converges.
4. You can use this fact to show that $n^{1/50} > \ln n$ for n large, so that $n^{1/2} > (\ln n)^{25}$ for n large enough.
5. Use the integral test to analyze this series.
6. We know that $\sin(1/n)$ goes to zero, but we actually know more than that. From calculus 1, we know that $\lim_{n\to\infty} n\sin(1/n) = 1$ because $\displaystyle\lim_{x\to 0}\frac{\sin x}{x} = 1$.

Hint #3

1. Use the substitution $u = x^3 + 4$ and see how that helps.
2. If we remove the non-dominant terms from the numerator and denominator, we can use the fact that $n^2 + 3n + 2 > n^2$ and $n^3 - 3 < n^3$ to give that $\dfrac{n^2 + 3n + 2}{n^3 - 3} > \dfrac{n^2}{n^3} = \dfrac{1}{n}$.
3. Look at $L = \lim\limits_{n\to\infty} \dfrac{a_n}{b_n}$ and see what this gives you.
4. This fact allows us to rewrite $\dfrac{(\ln(n))^{25}}{n^2}$ as $\dfrac{1}{n^{3/2}} \dfrac{(\ln(n))^{25}}{n^{1/2}}$, and for n large, this last factor is less than 1.
5. With the substitution $u = \ln(n)$, the integral converges, and so the series converges.
6. Since we know things about limits, we should try the limit comparison test here. Since $\lim_{n\to\infty} n \sin(1/n) = 1$, we know that $\lim n \to \infty \dfrac{\sin(1/n)}{\sqrt{n}} = \lim\limits_{n\to\infty} \dfrac{n\sin(1/n)}{n^{3/2}}$ which behaves like $\dfrac{1}{n^{3/2}}$.

Hint #4

2. What do we know about the series for $\frac{1}{n}$?

3. Since $L = 1$ what does that mean for the series?

4. Therefore, we know that for n large enough, $\dfrac{(\ln(n))^{25}}{n^2} < \dfrac{1}{n^{3/2}}$, and we know what happens to that series.

5. Now we can go back to the original series. How can we compare this series to the one we already know converges? Does the inequality go in the right direction to use direct comparison?

6. Do a limit comparison test with $\frac{1}{n^{3/2}}$ to see what you get. The results in the previous hint tells you the limit here is 1.

Answers

1. Since $\int_4^\infty \frac{x^2}{(x^3+4)^5}\,dx$ converges, the series $\sum_{n=4}^\infty \frac{n^2}{(n^3+4)^5}$ converges.

2. Since $\frac{n^2+3n+2}{n^3-3} > \frac{n^2}{n^3} = \frac{1}{n}$ and $\sum_{n=3}^\infty \frac{1}{n}$ diverges, we know that $\sum_{n=3}^\infty \frac{n^2+3n+2}{n^3-3}$ diverges.

3. By the Limit Comparison Test with the series $\sum_{n=2}^\infty \frac{1}{n^2}$, we see that $\sum_{n=2}^\infty \frac{n^4-3n^2+10}{(n^3+3n^2+2n+1)^2}$ converges.

4. By the Direct Comparison Test with the series $\sum_{n=1}^\infty \frac{1}{n^{3/2}}$, the series $\sum_{n=1}^\infty \frac{(\ln(n))^{25}}{n^2}$ converges.

5. Using the Limit Comparison test with $\sum_{n=3}^\infty \frac{1}{n(\ln(n))^2}$ (with limit $L=1$), we see that $\sum_{n=3}^\infty \frac{1}{n(\ln(n))^2-n}$ converges.

6. By limit comparison test with the series $b_n = \frac{1}{n^{3/2}}$, we see that the series $\sum_{n=6}^\infty \frac{\sin(1/n)}{\sqrt{n}}$ converges.