Math 152 - Worksheet 18

Power Series

Learning Problems

These problems should be completed on your own. If you need hints on solving a problem, there are some provided with each problem. These are provided on the following pages, with one 'level' of hint per page, with the earlier ones giving away less of the problem than the later ones. Try to work from the earlier hints to the later ones, as this will give you the practice you need to succeed in this class.

1. Determine for which values of x the series $\sum_{n=0}^{\infty} \frac{x^n}{n^2}$ converges.

2. Determine for which values of x the series $\sum_{n=0}^{\infty} \left(\frac{4}{3}\right)^n (x-2)^n$ converges.

3. Determine for which values of x the series $\sum_{n=0}^{\infty} \frac{1}{n2^n}(x+1)^n$ converges.

4. Determine for which values of x the series $\sum_{n=0}^{\infty} \frac{n}{5^n}x^{2n}$ converges.

5. Find a power series expansion for the function $\frac{1}{4+3x}$ and determine the interval of convergence.

6. Find a power series expansion for the function $\frac{2}{5-x}$ centered at $x=-1$ and determine the interval where it is valid.

Submission Problems

1. Determine for which values of x the series $\sum_{n=0}^{\infty} \frac{1}{n!}(x+2)^n$ converges.

2. Find a power series expansion for the function $\frac{2x}{(1-3x^2)^2}$ centered around $x=0$ and determine the interval of convergence. Hint: This is the derivative of a function with a nice power series expansion.

Hint #1

1. Use the Ratio Test to determine what the radius of convergence should be for this series.

2. Use the ratio test to determine what the radius of convergence should be.

3. Use the ratio test to try to figure out the radius of convergence.

4. Use the ratio test to try to figure out the radius of convergence. Be careful with the x^{2n} when working with the ratio test.

5. To get a power series expansion, we need to write this function in the form $\frac{1}{1-u}$ for some u.

6. The main trick here is that we know the power series for $\frac{1}{1-u}$ centered at 0. Since $\dfrac{1}{1-u} = \displaystyle\sum_{n=0}^{\infty} u^n$, this means that, for example, $\dfrac{1}{1-(x+1)} = \displaystyle\sum_{n=0}^{\infty}(x+1)^n$, which is a power series centered at -1.

Hint #2

1. The radius of convergence here is $R = 1$, so we know for sure that the series converges absolutely for $|x| < 1$.
2. The radius of convergence is $R = \frac{3}{4}$ so this series converges when $|x - 2| < \frac{3}{4}$.
3. The radius of convergence is 2 here, so we know the series for $|x + 1| < 2$.
4. The radius of convergence is $\sqrt{5}$ here, so we know the series for $|x| < \sqrt{5}$.
5. The function can be rewritten as $\dfrac{1}{4}\dfrac{1}{1 - (-\frac{3}{4}x)}$.
6. This means the expression we are aiming for is of the form $\dfrac{1}{1 - a(x + 1)}$ for some constant a. This will allow the power series to be centered at -1.

Hint #3

1. Next, we have to check the two endpoints at $x = 1$ and $x = -1$. For these, we get the series $\sum_{n=0}^{\infty} \frac{1}{n^2}$ and $\sum_{n=0}^{\infty} \frac{(-1)^n}{n^2}$. Do these converge?

2. Next, we need to check the endpoints. What happens if $x = 2 + \frac{3}{4}$ and $x = 2 - \frac{3}{4}$?

3. Next, we need to check the endpoints, which are at $x = -3$ and $x = 1$. What do these series look like?

4. Next, we need to check the endpoints, which are at $x = -\sqrt{5}$ and $x = \sqrt{5}$. What do these series look like?

5. Thus, the power series for this function is $\frac{1}{4} \sum_{n=0}^{\infty} \left(-\frac{3}{4}x\right)^n$

6. We can rewrite $\frac{2}{5-x} = 2\frac{1}{6-(x+1)} = \frac{1}{3}\frac{1}{1-\frac{1}{3}(x+1)}$.

Hint #4

1. Both of these series converge absolutely.

2. Those series become $\sum_{n=0}^{\infty} 1^n$ and $\sum_{n=0}^{\infty} (-1)^n$, both of which diverge.

3. The series we need to deal with are $\sum_{n=0}^{\infty} \frac{1}{n}$ and $\sum_{n=0}^{\infty} \frac{(-1)^n}{n}$

4. The series we need to deal with are $\sum_{n=0}^{\infty} n$ and $\sum_{n=0}^{\infty} n(-1)^n$

5. For finding the interval of convergence, we know this series converges whenever $|u| < 1$, and here, we have $u = -\frac{3}{4}x$

6. Thus, the power series we get is $\frac{1}{3}\sum_{n=0}^{\infty} \left(\frac{1}{3}(x+1)\right)^n$ and it is valid where $\left|\frac{1}{3}(x+1)\right| < 1$.

Answers

1. The interval of convergence here is $[-1, 1]$.
2. The interval of convergence here is $\left(\frac{5}{4}, \frac{7}{4}\right)$.
3. The interval of convergence here is $[-3, 1)$.
4. The interval of convergence here is $(-\sqrt{5}, \sqrt{5})$.
5. The power series expansion is $\displaystyle \frac{1}{4}\sum_{n=0}^{\infty}\left(-\frac{3}{4}\right)^n x^n$, and is valid for $|x| < \frac{4}{3}$.
6. The power series is $\displaystyle \frac{1}{3}\sum_{n=0}^{\infty}\frac{1}{3^n}(x+1)^n$ and is valid on $(-4, 2)$.