Math 152 - Worksheet 25

Area and Arc Length in Polar Coordinates

Learning Problems

These problems should be completed on your own. If you need hints on solving a problem, there are some provided with each problem. These are provided on the following pages, with one 'level' of hint per page, with the earlier ones giving away less of the problem than the later ones. Try to work from the earlier hints to the later ones, as this will give you the practice you need to succeed in this class.

1. Find the area of the triangle bounded by the x-axis, the y-axis, and the line $r = 4\sec\left(\theta - \frac{\pi}{4}\right)$ using an integral in polar coordinates. Then, check your answer using geometry.

2. Find the area inside one petal of the curve $r = \sin(4\theta)$.

3. Find the area inside the curve $r = 2 + \sin(2\theta)$ and outside the curve $r = \sin(2\theta)$.

4. Find the length of the curve $r = e^{\theta}$ for $0 \leq \theta \leq 2\pi$

5. Find the length of the curve $r = 1 + \theta$ over the range $0 \leq \theta \leq \pi$.

Submission Problems

1. Find the area of the region inside the outer loop but outside the inner loop of the function $r = 2\cos(\theta) - 1$.

2. Set up an integral (but do not evaluate) for the length of the polar curve $r = \sin^3(\theta)$ over the range $0 \leq \theta \leq \pi$.

Hint #1

1. Drawing out the picture shows that this line is in the first quadrant, so the boundaries are at $\theta = 0$ and $\theta = \frac{\pi}{2}$.

2. First, we need to figure out where a petal starts and stops. Where does this curve come back to the origin?

3. Since $\sin(2\theta)$ is always between -1 and 1, we know that the curve $r = 2 + \sin(2\theta)$ is always outside $r = \sin(2\theta)$. Thus, we just need to find the area inside each curve and subtract them.

4. We have a standard formula for the length of a polar curve that we can use to find this length.

5. Again, this goes back to the standard formula for length of polar curves.

Hint #2

1. The integral we get here is $\frac{1}{2}\int_0^{\pi/2} (4\sec\left(\theta - \frac{\pi}{4}\right)^2 \, d\theta$

2. This starts at the origin at $\theta = 0$, and the first time it comes back is when $4\theta = \pi$, or $\theta = \frac{\pi}{4}$.

3. For $2 + \sin(2\theta)$, the radius is always positive, so there aren't any petals. Thus, to find the area, we just need to integrate from 0 to 2π.

4. Since our function is $f(\theta) = e^\theta$, then our integral becomes $\int_0^{2\pi} \sqrt{f(\theta)^2 + f'(\theta)^2} \, d\theta = \int_0^{2\pi} \sqrt{e^{2\theta} + e^{2\theta}} \, d\theta$.

5. Since our function is $f(\theta) = 1 + \theta$, then our integral becomes $\int_0^{\pi} \sqrt{f(\theta)^2 + f'(\theta)^2} \, d\theta = \int_0^{\pi} \sqrt{(1+\theta)^2 + 1} \, d\theta$.

Hint #3

1. Using a substitution $u = \theta - \frac{\pi}{4}$, the integral becomes $8\tan\left(\theta - \frac{\pi}{4}\right)$ from 0 to $\pi/2$, which comes out to 16.

2. Thus, the integral we need here is $\displaystyle\int_0^{\pi/4} \frac{1}{2}(\sin(4\theta))^2 \, d\theta$.

3. For the inside curve, the radius does go to zero, so there are petals. By plotting the curve, we know that there are 4 petals, and the first one goes up to $\theta = \frac{\pi}{2}$.

4. This integral simplifies to $\displaystyle\int_0^{2\pi} \sqrt{2}e^{\theta} \, d\theta$

5. This integral can be left as is, and use $1 + \theta = \tan(\phi)$ as a trigonometric substitution. This will turn into a reduction formula, needing to integrate $\sec^3\phi$.

Hint #4

1. For this triangle, the point on the x axis is at $\theta = 0$, where $r = 4\sec -\frac{\pi}{4} = 4\sqrt{2}$, and the same holds for the point along the y axis. Thus, the area is $\frac{1}{2}(4\sqrt{2})(4\sqrt{2}) = 16$

2. This integral can be evaluated using our half angle formulas from before.

3. The outside area is $\dfrac{1}{2}\displaystyle\int_0^{2\pi}(2+\sin(2\theta))^2\,d\theta$, and the inside area is four times $\dfrac{1}{2}\displaystyle\int_0^{\pi/2}(\sin(2\theta))^2\,d\theta$.

5. Integrating this out, we get that $\displaystyle\int \sec^3\phi d\phi = \frac{1}{2}\sec\phi\tan\phi + \frac{1}{2}\ln|\sec\phi + \tan\phi|$. This can then be converged back to θ, and then plug in 0 and π

Answers

1. 16
2. $\dfrac{\pi}{16}$
3. $\dfrac{9\pi}{2} - \dfrac{\pi}{2} = 4\pi$
4. $\sqrt{2}(e^{2\pi} - 1) = 755.89$
5. $\left(\dfrac{1}{2}(1+\theta)(\sqrt{\theta^2 + 2\theta + 2} + \dfrac{1}{2}\ln(\sqrt{\theta^2 + 2\theta + 2} + 1 + \theta)\right) |_0^\pi = \dfrac{1}{2}\left((\pi + 1)\sqrt{\pi^2 + 2\pi + 2} + \ln(1 + \pi + \sqrt{\pi\right.$
8.739