# Math 152 - Worksheet 23

Arc Length and Speed

## Learning Problems

These problems should be completed on your own. If you need hints on solving a problem, there are some provided with each problem. These are provided on the following pages, with one 'level' of hint per page, with the earlier ones giving away less of the problem than the later ones. Try to work from the earlier hints to the later ones, as this will give you the practice you need to succeed in this class.

1. Find the length of the path $c(t) = (4t, 2t^{3/2})$ between $t = 1$ and $t = 3$.
2. Find the length of the spiral $c(t) = (t\cos(t), t\sin(t))$ between $t = 0$ and $t = 4\pi$.
3. Find the minimum speed of the particle whose trajectory is given by $c(t) = (2t^3, t^{-2})$ with $t \geq 0.5$ for $t$ in seconds and $c(t)$ in meters.
4. Find the surface area of the solid of revolution generated by revolving the curve $c(t) = (3t^2, 2t)$ between $t = 1$ and $t = 3$ around the $x$-axis.

## Submission Problems

1. Find the length of the path $c(t) = (t^3 + 1, t^2 - 3)$ between $t = 0$ and $t = 4$.
2. Find the surface area of the solid of revolution generated by revolving the curve $c(t) = (\sin^2(t), \cos^2(t)$ between $t = 0$ and $t = \pi/2$ around the $x$-axis.

## Hint #1

1. We can use the standard arc length formula here to find this length.

2. We can again apply the standard formula here, with $x'(t) = \cos(t) - t\sin(t)$ and $y'(t) = \sin(t) + t\cos(t)$.

3. To find the minimum speed, we need to first get an equation for the speed. How do we get this?

4. We have the standard formula for surface area here: $2\pi \int_a^b y(t)\sqrt{x'(t)^2 + y'(t)^2}\, dt$.

## Hint #2

1. Since $x'(t) = 4$ and $y'(t) = 3t^{1/2}$, we know that the integral we need to compute is $\int_1^3 \sqrt{4^2 + (3\sqrt{t})^2}\, dt$

2. Expanding and simplifying gives that $x'(t)^2 + y'(t)^2 = 1 + t^2$. Work this out for yourself, it relies on the fact that $\sin^2(t) + \cos^2(t) = 1$.

3. The speed is $\sqrt{x'(t)^2 + y'(t)^2}$, which for this curve is $\sqrt{(6t^2)^2 + (-2t^{-3})^2}$

4. Plugging in the different components gives that we need to compute $2\pi \int_1^3 2t\sqrt{36t^2 + 4}\, dt$.

## Hint #3

1. This integral can be evaluated by substitution with $u = 16 + 9t$.
2. Thus, we need to compute $\int_0^{4\pi} \sqrt{1+t^2}\, dt$. This will require trigonometric substitution and then reduction formulas.
3. To find the minimum of this, we can instead look for the minimum of the square of the speed, because that will be simpler. Thus, we need to find the minimum of the function $36t^4 + 4t^{-6}$.
4. This can be solved by substitution, using $u = 4 + 36t^2$, so that $du = 72t\ dt$.

## Hint #4

2. With $t = \tan\theta$, this integral becomes $\int \sec^3\theta d\theta$, which can be integrated by reduction formulas.

3. The derivative of this function is $144t^3 - 24t^{-7}$, which is equal to zero at $t = 6^{-1/10}$

4. The integral then becomes $2\pi \int_{40}^{328} \frac{1}{36}\sqrt{u}\, du$.

## Answers

1. $\displaystyle\int_1^3 \sqrt{16+9t}\ dt = \frac{1}{9}\int_{25}^{43} \sqrt{u}\ du = \frac{2}{27}\left((43)^{3/2} - (25)^{3/2}\right) = 11.627$

2. 80.82

3. The minimum speed is the speed at $6^{-1/10}$, which is 29.3023.

4. $\displaystyle\frac{\pi}{27}\left((328)^{3/2} - (40)^{3/2}\right) = 661.75$