# Math 152 - Worksheet 21

Arc Length and Surface Area

## Learning Problems

These problems should be completed on your own. If you need hints on solving a problem, there are some provided with each problem. These are provided on the following pages, with one 'level' of hint per page, with the earlier ones giving away less of the problem than the later ones. Try to work from the earlier hints to the later ones, as this will give you the practice you need to succeed in this class.

1. Compute the length of the graph of $f(x) = 9 - 4x$ between $x = 1$ and $x = 5$.
2. Compute the length of the graph of $f(x) = \frac{x^4}{16} + \frac{1}{2x^2}$ over the interval $[2, 6]$
3. Compute the length of the graph of $f(x) = \ln\left(\dfrac{e^x + 1}{e^x - 1}\right)$ over the interval $[2, 4]$.
4. Compute the surface area of the solid of revolution obtained by revolving the graph of $y = \cos x$ between $x = 0$ and $x = \frac{\pi}{2}$ around the $x$-axis.

## Submission Problems

1. Find the arc length of the curve $y = \sqrt{4 - x^2}$ between $x = -1$ and $x = 1$. Compute this using normal integration formulas, and check your answer using geometry.
2. Find the surface area of the solid of revolution formed by rotating the curve $y = x^2$ between $x = 1$ and $x = 5$ around the $x$-axis.

## Hint #1

1. What is our formula for length of curves?
2. For our formula, we need $f'(x)$. What is $f'$ here and what does $1 + f'(x)^2$ look like?
3. For this function $f'(x) = \dfrac{e^x - 1}{e^x + 1} \cdot \dfrac{(e^x - 1)(e^x) - (e^x + 1)(e^x)}{(e^x - 1)^2} = \dfrac{-2e^x}{(e^x - 1)(e^x + 1)} = \dfrac{-2e^x}{e^{2x} - 1}$.
4. What is the formula for computing the surface area of a solid of revolution?

## Hint #2

1. For this problem, we need to find $\int_1^5 \sqrt{1+f'(x)^2}\,dx$, where $f'(x) = -4$ here.

2. $f'(x) = \dfrac{x^3}{4} - \dfrac{1}{x^3}$, so $1 + f'(x)^2 = 1 + \dfrac{x^6}{16} - \dfrac{1}{2} + \dfrac{1}{x^6} = \dfrac{x^6}{16} + \dfrac{1}{2} + \dfrac{1}{x^6}$

3. Taking this, we can simplify $1 + f'(x)^2$ to get a perfect square. This simplification gives $1 + \dfrac{4e^{2x}}{(e^{2x}-1)^2} = \dfrac{e^{4x} - 2e^{2x} + 1 + 4e^{2x}}{(e^{2x}-1)^2} = \dfrac{(e^{2x}+1)^2}{(e^{2x}-1)^2}$ so that we need to integrate $\displaystyle\int_2^4 \frac{e^{2x}+1}{e^{2x}-1}\,dx$.

4. Since the formula is $\int_a^b 2\pi f(x)\sqrt{1+f'(x)^2}\,dx$, we need to compute $\int_0^{\pi/2} 2\pi\cos(x)\sqrt{1+\sin^2 x}\,dx$.

## Hint #3

1. Thus, we need to find $\int_1^5 \sqrt{5}\ dx$.

2. In this case $1 + f'(x)^2$ is a perfect square, so $1 + f'(x)^2 = \left(\frac{x^3}{4} + \frac{1}{x^3}\right)^2$, which makes it easy to work out the arc length integral.

3. Using the substitution $u = e^{2x} - 1$ gives $du = 2e^{2x}\ dx = 2(u+1)\ dx$. This gives the integral we need to compute as $\int_2^4 \frac{1}{2} \frac{u+2}{u} \frac{du}{u+1}$, which we can solve by partial fractions.

4. We can evaluate this integral by setting $u = \sin x$, so $du = \cos x\ dx$. This turns the integral into $2\pi \int_0^1 \sqrt{1+u^2}\ du$.

## Hint #4

1. You can check the answer here by computing the hypotenuse of a right triangle.
2. For this answer, we need to compute $\int_2^6 \frac{x^3}{4} + \frac{1}{x^3}\, dx$
3. Since $\frac{u+2}{u(u+1)} = \frac{2}{u} - \frac{1}{u+1}$, we can evaluate this integral from $e^4 - 1$ to $e^8 - 1$.
4. This now requires trig substitution for $u = \tan(\theta)$ and then a reduction formula to get to the final answer here.

## Answers

1. $4\sqrt{5}$

2. $\dfrac{721}{9} = 80.111$

3. $\dfrac{1}{2}\left(2\ln e^8 - 1 - 2\ln e^4 - 1 - \ln e^8 + \ln e^4\right) = \ln\left(\dfrac{e^8-1}{e^4-1}\right) - 2 = 2.0181$

4. $1.1478$