Complex Numbers Part 2

Learning Goals

- Compute powers of complex numbers using the exponential form
- Compute all of the nth roots of a complex number
- Use De Moivre's Formula to find formulas for $\cos(n\theta)$ and $\sin(n\theta)$.
- Find the nth roots of unity.
- Find the complex roots of polynomials.

Contents

1 Powers of Complex Numbers

Multiplying complex numbers is really easy in polar form.

$$re^{i\theta} \cdot se^{i\varphi} = (rs)e^{i(\theta+\varphi)}$$

What does this mean for taking powers of complex numbers?

$$z = re^{i\theta}$$

$$z^2 = re^{i\theta} \cdot re^{i\theta} = r^2 e^{i(\theta+\theta)}$$
$$= r^2 e^{i(2\theta)}$$

$$z^4 = r^4 e^{i(4\theta)}$$

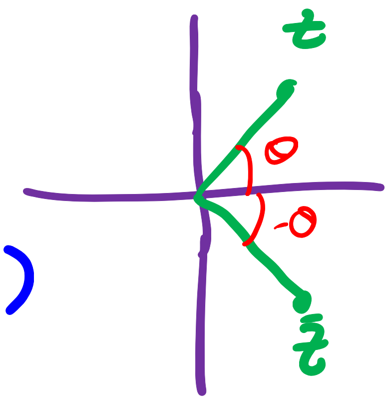
For any positive n,

$$\boxed{z^n = r^n e^{in\theta} \quad \text{for } z = re^{i\theta}}$$

What about negative powers of complex numbers?

- These work in the same way!

$$z = re^{i\theta} = r\cos\theta + i\, r\sin\theta$$



$$\bar{z} = r\cos\theta - i\, r\sin\theta$$

$$= r\cos(-\theta) + i\, r\sin(-\theta)$$

$$= re^{-i\theta}$$

$$\frac{1}{z} = \frac{\bar{z}}{|z|^2} = \frac{re^{-i\theta}}{r^2} = \frac{1}{r}e^{-i\theta} =$$

$$= r^{-1}e^{i(-1)\theta}$$

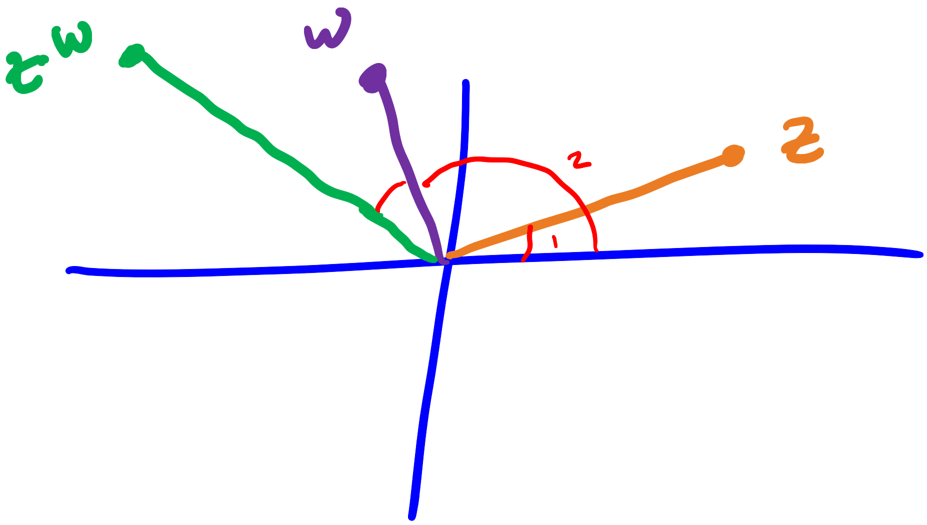
$$z^{-2} = r^{-2}e^{i(-2)\theta}$$

For any integer n $\boxed{z^n = r^n e^{in\theta}}$

Geometry of Powers

$$re^{i\theta} \cdot se^{i\varphi} = rs\, e^{i(\theta+\varphi)}$$

- Multiply Moduli
- Add angles of rotation.



For powers of $z = re^{i\theta}$

- Modulus is the power of the original.
- Angle is n-multiples of the original angle from z

Example: Compute z^{-1}, z^2, z^3, and z^4 for the complex number $z = 1 - \sqrt{3}i$.

Polar

$$r = \sqrt{1^2 + \sqrt{3}^2} = \sqrt{4} = 2$$

$$\theta = \tan^{-1}\left(\frac{-\sqrt{3}}{1}\right) = -\pi/3$$

$$\boxed{\begin{aligned}
z &= 2e^{-i\pi/3} &&= 1 - \sqrt{3}i \\
z^{-1} &= \frac{1}{2}e^{i\pi/3} &&= \frac{1}{4} + \frac{\sqrt{3}}{4}i \\
z^2 &= 4e^{-i2\pi/3} &&= -2 - 2\sqrt{3}i \\
z^3 &= 8e^{-i\pi} &&= -8 \\
z^4 &= 16e^{-i4\pi/3} &&= -8 + 8\sqrt{3}i
\end{aligned}}$$

2 Roots of Complex Numbers

Take a complex number $w = re^{i\theta}$, and say that we want to solve the equation $z^n = w$. How could we do this?

$z = se^{i\varphi}$

$z^n = s^n e^{in\varphi} = w = re^{i\theta}$

$s^n = r$ $\quad n\varphi = \theta$

$s = r^{1/n} = \sqrt[n]{r}$ $\quad \varphi = \frac{\theta}{n}$

$z = r^{1/n} e^{i\theta/n}$

satisfies

$z^n = w = re^{i\theta}$

What other values of z are there?

$w = re^{i\theta}$ or (r, θ) in polar coordinates.

$$(r, \theta) = (r, \theta + 2\pi)$$
$$= (r, \theta + 4\pi)$$
$$\vdots$$
$$= (r, \theta + 2k\pi)$$ for any integer k.

$w = re^{i(\theta + 2\pi)}$

$z^n = w$

$z = r^{1/n} e^{i\left(\frac{\theta}{n} + \frac{2\pi}{n}\right)}$

$\rightarrow$ Different complex number than $r^{1/n} e^{i\theta/n}$

$z = r^{1/n} e^{i\left(\frac{\theta}{n} + \frac{4\pi}{n}\right)}$

General Formulas

$w = r e^{i(\theta + 2\pi k)}$ $\quad k = 0, \ldots, n-1$

For the complex number $w = re^{i\theta}$, and a positive number n, the solutions to the equation $z^n = w$ are

$$z_0 = r^{1/n} e^{i\theta/n}$$

$$z_1 = r^{1/n} e^{i\left(\frac{\theta}{n} + \frac{2\pi}{n}\right)}$$

$$z_2 = r^{1/n} e^{i\left(\frac{\theta}{n} + \frac{4\pi}{n}\right)}$$

$$\vdots$$

$$z_{n-1} = r^{1/n} e^{i\left(\frac{\theta}{n} + \frac{2\pi(n-1)}{n}\right)}$$

$$\boxed{z_k = r^{1/n} e^{i\left(\frac{\theta + 2\pi k}{n}\right)}}$$

Example: Find the polar coordinates of the 3 complex solutions of $z^3 = 1+i$.

$w = 1+i \qquad n = 3$

w into polar

$r = \sqrt{1^2+1^2} = \sqrt{2}$

$\theta = \tan^{-1}\left(\frac{1}{1}\right) = \pi/4$

$2^{1/2}$

$w = \sqrt{2}\, e^{i\pi/4}$

$$\boxed{\begin{aligned} z_1 &= 2^{1/6} e^{i\pi/12} \\ z_2 &= 2^{1/6} e^{i\left(\frac{\pi/4+2\pi}{3}\right)} = 2^{1/6} e^{i 9\pi/12} \\ z_3 &= 2^{1/6} e^{i\left(\frac{\pi/4+4\pi}{3}\right)} = 2^{1/6} e^{i 17\pi/12} \end{aligned}}$$

3 Roots of Unity

An important result of roots of complex numbers is the idea of roots of unity.

Definition. The **nth roots of unity** are the n complex solutions to the equation $z^n = 1$.

- Raising to the n^{th} power gives 1.

$z = 1$ $\qquad$ $1 = 1e^{i0}$

$$z_k = 1e^{i\left(\frac{0+2k\pi}{n}\right)}$$

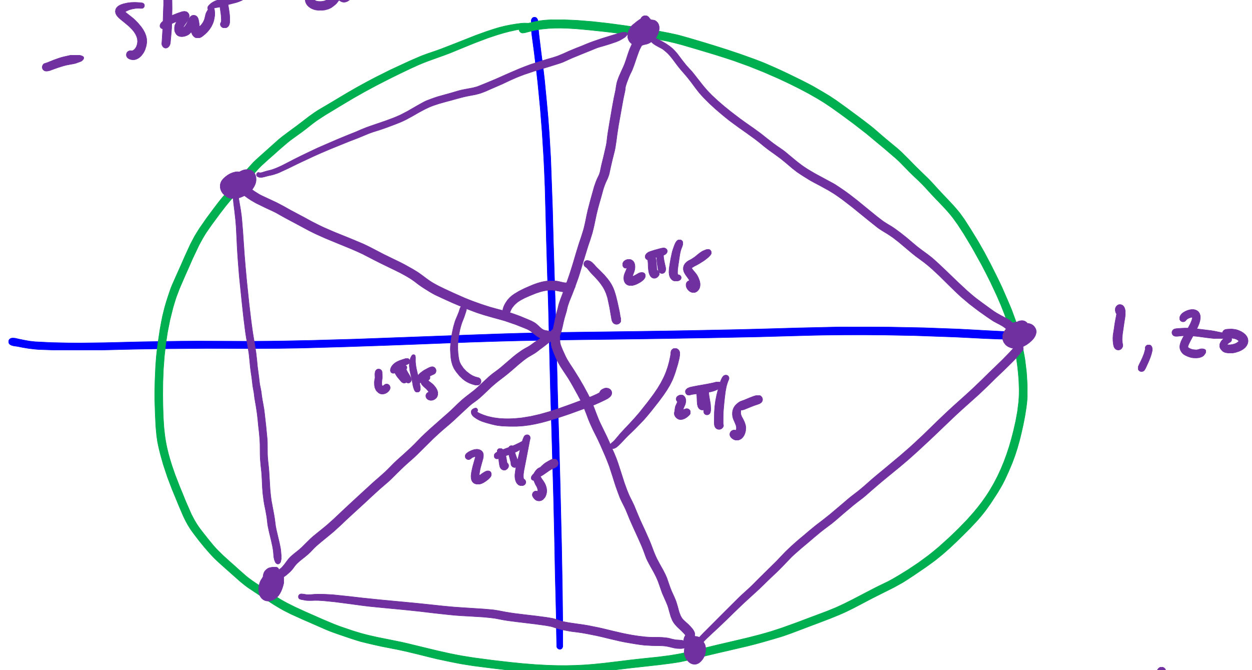
$$\boxed{z_k = e^{i\,2k\pi/n}}$$

$k = 0, 1, 2, \ldots n-1,$

What does this look like plotted out?

$$e^{i\frac{2\pi k}{n}}$$

- Modulus $= 1$
- Angles are $\frac{2\pi k}{n}$
  - Evenly spaced, $\frac{2\pi}{n}$ between them
  - Start at $\theta = 0$



→ Regular n-gon out of this.

$e^{i\frac{2k\pi}{6}}$

Example: Find the polar and rectangular coordinates of the 6th roots of unity.

$z_0 = e^{i\cdot 0} = 1$

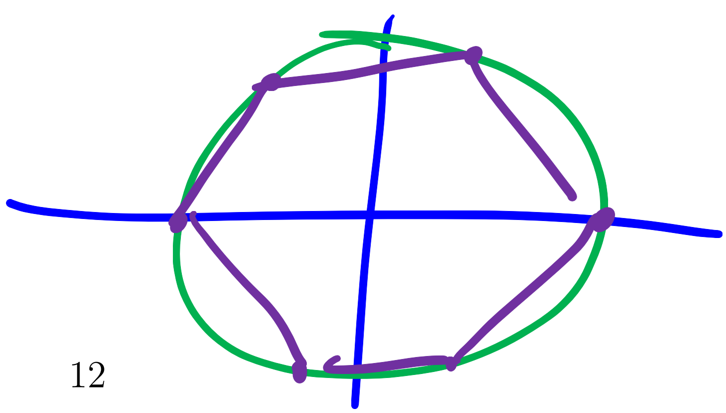
$z_1 = e^{i\pi/3} = \frac{1}{2} + \frac{\sqrt{3}}{2}i$

$z_2 = e^{i2\pi/3} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$

$z_3 = e^{i\pi} = -1$

$z_4 = e^{i4\pi/3} = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$

$z_5 = e^{i5\pi/3} = 1/2 - \frac{\sqrt{3}}{2}i$



4 De Moivre's Formulas

We have easy formulas for the powers of complex numbers. What else does that give us?

$$z = re^{i\theta}$$

$$z^n = r^n e^{in\theta}$$

$$\left(e^{i\theta}\right)^n = \left(\cos\theta + i\sin\theta\right)^n$$

$$e^{in\theta} = \left(\cos\theta + i\sin\theta\right)^n$$

$$\boxed{\cos(n\theta) + i\sin(n\theta) = \left(\cos\theta + i\sin\theta\right)^n}$$

De Moivre's Formulas

For any θ and any positive integer n, we have that

$$\boxed{\begin{aligned} \cos(n\theta) + i\sin(n\theta) &= (\cos\theta + i\sin\theta)^{n} \\ \cos(n\theta) - i\sin(n\theta) &= (\cos\theta + i\sin\theta)^{-n} \end{aligned}}$$

Example: Use De Moivre's Formulas to derive the double angle formulas for sine and cosine.

$$n=2$$

$$\cos(2\theta) + i\sin(2\theta) = (\cos\theta + i\sin\theta)^2$$

$$\cos 2\theta + i\sin 2\theta = \cos^2\theta + 2i\sin\theta\cos\theta + i^2\sin^2\theta$$

$$\underbrace{\cos(2\theta)} + i\underbrace{\sin 2\theta} = \underbrace{(\cos^2\theta - \sin^2\theta)} + 2i\underbrace{\sin\theta\cos\theta}$$

$$\boxed{\begin{aligned}\cos(2\theta) &= \cos^2\theta - \sin^2\theta \\ \sin 2\theta &= 2\sin\theta\cos\theta\end{aligned}}$$

5 Polynomials and Complex Numbers

The main reason we care about complex numbers is the Fundamental Theorem of Algebra:

Theorem. *Any polynomial equation of the form $P(z) = 0$ of degree $n \geq 1$ has exactly n complex solutions, counting repeated roots.*

What does this mean?

- "Not having solutions" doesn't exist for polynomials in terms of complex numbers.

- Every quadratic always has two solutions.
  - Two distinct real solutions
  - Two complex solutions
  - One repeated root $[(x-4)^2]$

- There are 5 complex solutions to $x^5 - 4x^3 + 3x^2 = 4$

What other facts do we have?

- If $P(z)$ is a real polynomial
  (all coefficients are real numbers)
  Then complex roots come in conjugate pairs
  - If $a+bi$ is a root, so is $a-bi$

- Can't have 1 real and one complex (non real) root to a quadratic.

- Every cubic must have at least one real root.
  - If a cubic has two real roots, then it must have three real roots.

→ Can't have an odd number of non-real roots.

Example: Find all solutions of the equation $x^5 + z^3 - z^2 - 1. = 0$

$$x^3(x^2+1) - 1\,(x^2+1) = 0$$

$$(x^3-1)(x^2+1) = 0$$

$$x^2+1 = 0 \qquad x = i, \quad x = -i$$

$$x^3 - 1 = 0 \quad \text{or} \quad x^3 = 1$$

$$x = e^{i2\pi/3} \qquad x = e^{i4\pi/3} \qquad x = 1$$

Five Solutions: $\boxed{x = \{1, i, -i, e^{i2\pi/3}, e^{i4\pi/3}\}}$