Complex Numbers

Learning Goals

- Add, subtract, multiply, and divide complex numbers
- Find the absolute value of a complex number
- Convert complex numbers to and from polar form
- Find the product and quotient of complex numbers in polar form
- Use complex numbers to help solve partial fraction problems
- Use complex numbers to discuss the radius of convergence of power series

Contents

1 Algebra of Complex Numbers

Why Complex Numbers?

We know that the polynomial $x^2 + 1$ has no real roots. But what if it had roots?

$$x^2+1=0$$

$$x^2=-1 \quad \text{or} \quad x=\pm\sqrt{-1}$$

Define the imaginary unit i as $\sqrt{-1}$

- The solutions to $x^2+1=0$ are $\pm i$
- $\sqrt{-16} = \sqrt{-1}\cdot\sqrt{16} = \boxed{4i}$
- Solving Quadratic equations using this i.

Definition. A **complex number** x is defined as

$$z = x + iy \qquad i = \sqrt{-1}$$

for x and y real numbers. For this number x is the **real part** of z and y is the **imaginary part** of z.

$z = x + iy$

$x = \mathrm{Re}(z)$

$y = \mathrm{Im}(z)$

$z = 1 + 3i$

$\mathrm{Re}(z) = 1$

$\mathrm{Im}(z) = 3$

$w = \sqrt{2} - \frac{1}{3}i$

$\mathrm{Re}(w) = \sqrt{2}$

$\mathrm{Im}(w) = -1/3$

Operations on Complex Numbers

All of the operations we can do on real numbers we can also do with complex numbers. The idea is that we treat i like a variable and group terms so it matches the form of a complex number.

$$(a+bi) + (x+yi) = \underline{(a+x) + (b+y)i}$$

$$a + bi + x + yi$$

$$(a+bi) - (x+yi) = \underline{(a-x) + (b-y)i}$$

$$(a+bi)(x+yi) = ax + xbi + ayi + biyi$$

$$= ax + xbi + ayi + by\underbrace{i^2}_{-1}$$

$$= \boxed{ax - by + (xb + ay)i}$$

Example: Compute the following:

1. $(2+3i)-(5-4i)$
2. $3(2+i)+4(4-i)$
3. $(2+i)(3-2i)$

1. $(2-5)+(3i+4i)$

$$\boxed{=-3+7i}$$

2. $(6+3i)+(16-4i)$

$$(6+16)+(3-4)i = \boxed{22-i}$$

3. $6+3i-4i-2\underbrace{i^2}_{-1}$

$$6+3i-4i+2$$

$$\boxed{8-i}$$

2 Complex Conjugate and Division

The last operation on real numbers that we want to extend to complex numbers is division. How do we think about division of real numbers?

$$\frac{a}{b} = a \cdot \frac{1}{b}$$

Complex Numbers: What is $\frac{1}{z}$?

→ Division is multiplying by $1/z$.

$$1+i \to \frac{1}{1+i} \rightsquigarrow x+iy$$

Once it is in that form we're good.

For this, we need another definition:

Definition. The **complex conjugate** of $z = x + iy$ is the complex number $\bar{z} = x - iy$. The **modulus** of a complex number $z = x + iy$ is $|z| = \sqrt{x^2 + y^2}$.

$\bar{z} = x - iy$

$Re(z) = Re(\bar{z})$

$Im(z) = -Im(\bar{z})$

If $z = x$ is a real number, then

$|z| = \sqrt{x^2 + 0^2} = \sqrt{x^2} = |x|$

Modulus

Absolute value

$\bar{z} = 3 - i$

$z = 3 + i$

Properties of Complex Conjugates

(a) $\overline{\overline{z}}$ $= z$

(b) $\overline{z_1 + z_2}$ $= \overline{z_1} + \overline{z_2}$

(c) $\overline{z_1 \cdot z_2}$ $= \overline{z_1} \cdot \overline{z_2}$

(d) $z \cdot \overline{z}$ $(x+iy)(x-iy)$

$= x^2 + \cancel{iyx} - \cancel{xiy} - i^2y^2$

$= x^2 + y^2 = |z|^2$

Real Number

- Can divide by this and get a real number.

Reciprocal of Complex Numbers

If we look at the product $z \cdot \underbrace{\frac{\bar{z}}{|z|^2}}$, what do we get?

$$= \frac{z \cdot \bar{z}}{|z|^2} = \frac{|z|^2}{|z|^2} = 1$$

$$\boxed{\frac{\bar{z}}{|z|^2} = \frac{1}{z}}$$

Not in $x+iy$ form

This is!

So we can use this with multiplication to find the quotients we need.

Example: Find $\dfrac{2+3i}{1-i}$. $= (2+3i)\cdot\dfrac{1}{1-i}$

$$\frac{1}{1-i} = \frac{\bar{z}}{|z|^2}$$

$z = 1-i$

$\bar{z} = 1+i$

$|z|^2 = 1^2 + (-1)^2 = 2$

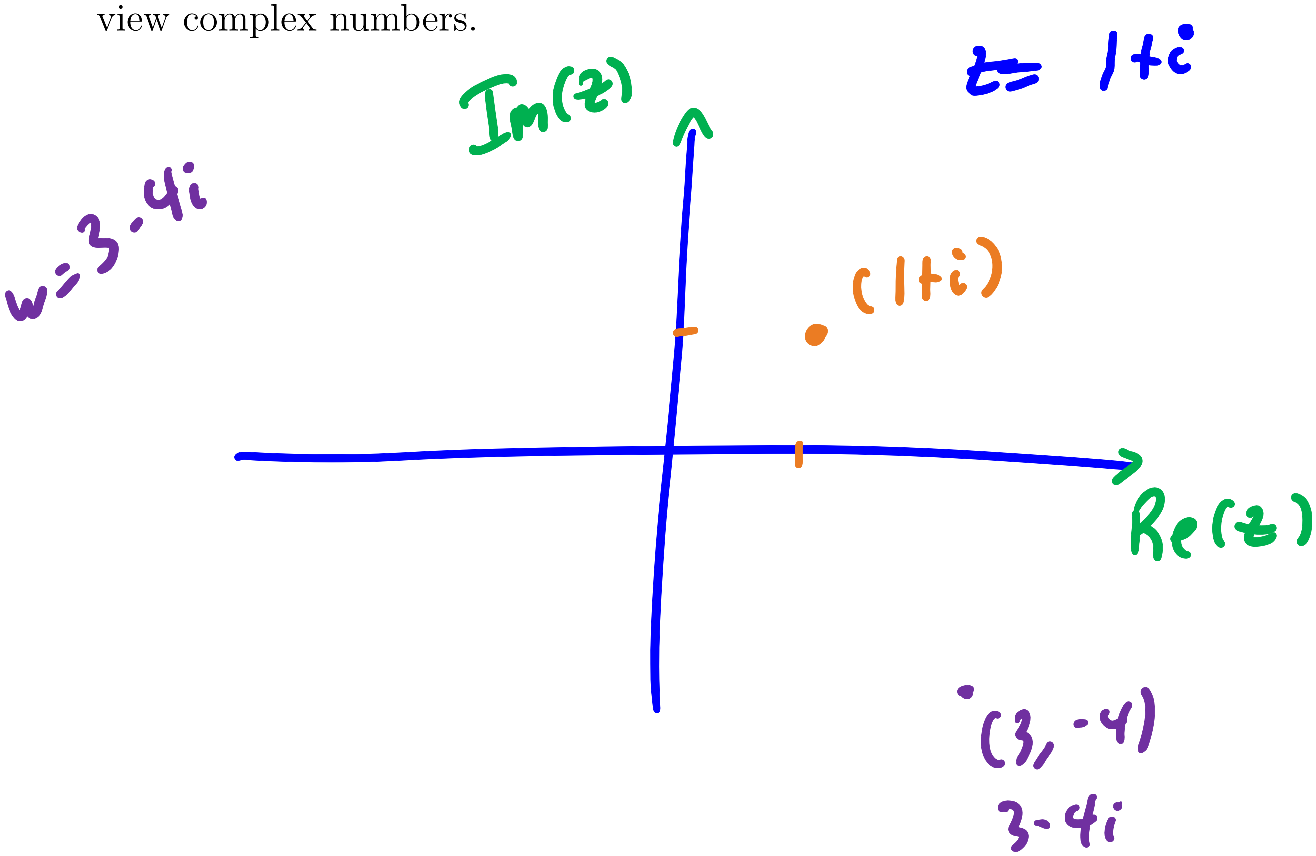
$$\frac{1}{1-i} = \frac{1+i}{2} = \frac{1}{2} + \frac{1}{2}i$$

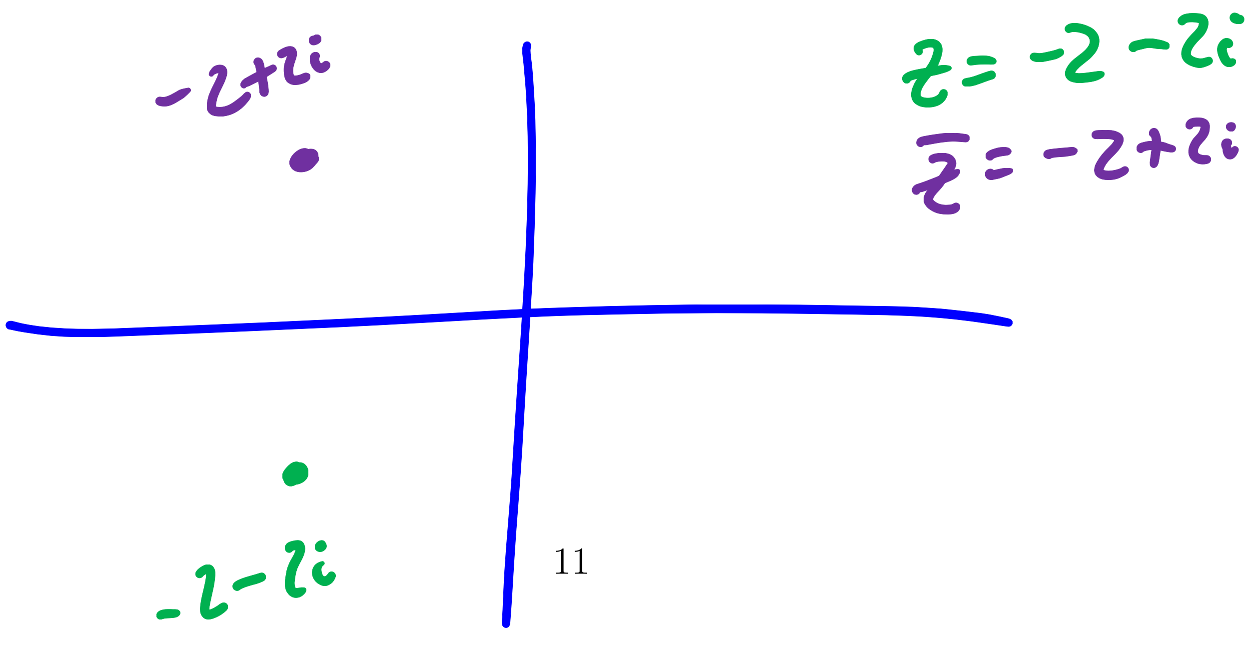
$$\frac{2+3i}{1-i} = (2+3i)\left(\frac{1}{2}+\frac{1}{2}i\right) = 1 + \frac{3}{2}i + i + \frac{3}{2}i^2$$

$$= \boxed{-\frac{1}{2} + \frac{5}{2}i}$$

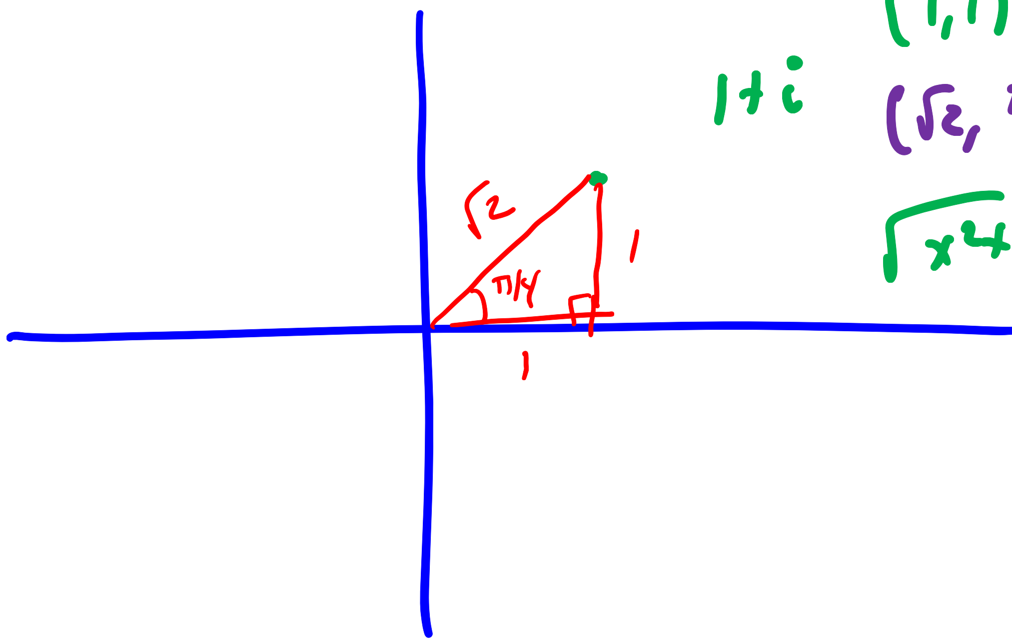
3 Geometry of Complex Numbers

How can we visualize complex numbers? The notation $z = x + iy$ is suggestive here, in that we can use the x and y coordinates in the plane to plot and view complex numbers.





We can also use polar coordinates to view these numbers.



$(1,1)$

$1+i$ $\quad (\sqrt{2}, \pi/4)$

$\sqrt{x^2+y^2} = r$

$= |z|$

$x + iy$

$r\cos\theta + i(r\sin\theta)$

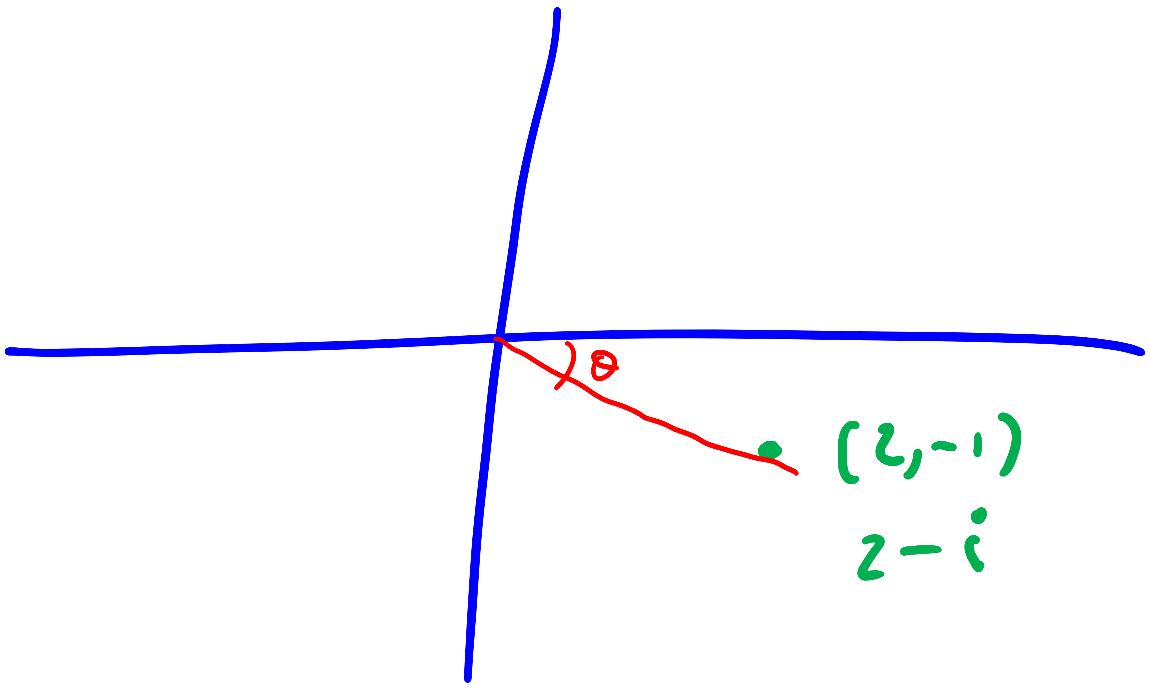
$$\boxed{r(\cos\theta + i\sin\theta)}$$

$\underline{r = |z|}$

$\sqrt{2}(\cos\pi/4 + i\sin\pi/4)$

$1 + i$

Example: Plot the complex number $2 - i$ in the complex plane as well as the polar coordinates of this number.



$$|z| = \sqrt{2^2 + (-1)^2} = \sqrt{5} = r$$

$$\theta = \tan^{-1}\left(\frac{-1}{2}\right) = \tan^{-1}\left(\frac{-1}{2}\right)$$

$$\sqrt{5}\left(\cos(\theta) + i\sin(\theta)\right)$$

4 Exponential Form and Euler's Formula

When we write a complex number in polar form, we see that it can be written as

$$z = |z|\cos\theta + i|z|\sin\theta = |z|\underbrace{(\cos\theta + i\sin\theta)}$$

Is there a nicer way to view this?

$$\cos\theta = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\theta^{2n} \qquad \sin\theta = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\theta^{2n+1}$$

$$-1 = i^2$$

$$\cos\theta = \sum_{n=0}^{\infty} \frac{(i^2)^n}{(2n)!}\theta^{2n} = \sum_{n=0}^{\infty} \frac{1}{(2n)!}(i\theta)^{2n}$$

$$\sin\theta = \sum_{n=0}^{\infty} \frac{(i^2)^n}{(2n+1)!}\theta^{2n+1} = \frac{1}{i}\sum_{n=0}^{\infty} \frac{1}{(2n+1)!}(i\theta)^{2n+1}$$

$$\cos\theta + i\sin\theta = \underbrace{\sum_{n=0}^{\infty} \frac{1}{(2n)!}(i\theta)^{2n}}_{\text{All even terms}} + \underbrace{\sum_{n=0}^{\infty} \frac{1}{(2n+1)!}(i\theta)^{2n+1}}_{\text{All odd terms.}}$$

$$\cos\theta + i\sin\theta = \underbrace{\sum_{n=0}^{\infty} \frac{1}{n!}(i\theta)^n}_{\text{Maclaurin Series for } e^x}$$

$$\boxed{\cos\theta + i\sin\theta = e^{i\theta}}$$

Euler's Formula

Definition. The **polar form** of a complex number z is $|z|e^{i\theta}$.

$$\underbrace{|z|}(\cos\theta + i\sin\theta)$$

- Leading coefficient must be positive.

$-1 = -1 + 0i$

$(-1, 0) \rightarrow \underset{}{\overset{\text{polar}}{(1, \pi)}}$

$$-1 = 1e^{i\pi} \rightarrow \boxed{e^{i\pi} + 1 = 0}$$

Product and Quotient in Polar Form

It is really easy to add and subtract complex numbers in rectangular form $x+iy$, and slightly more complicated to multiply and divide in this form. For polar form or exponential form, however, multiplying and dividing is really easy.

- All exponential Rules apply to complex Numbers.

$$e^{i\theta} \cdot e^{i\varphi} = e^{i(\theta+\varphi)}$$

$$2e^{i\pi/4} \cdot 3e^{-i\pi/6} = 6e^{i(\pi/4 - \pi/6)}$$

$$= 6e^{i\pi/12}$$

Example: Convert to exponential form and then find the quotient $\frac{3-3i}{1+\sqrt{3}i}$.

Polar Form

$3-3i$

$$|z| = \sqrt{9+9} = 3\sqrt{2}$$

$$\theta = \tan^{-1}\left(\frac{-3}{3}\right) = -\pi/4$$

$1+\sqrt{3}i$

$$|z| = \sqrt{1+3} = 2$$

$$\theta = \tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = \pi/3$$

$$\frac{3-3i}{1+\sqrt{3}i} = \frac{3\sqrt{2}\,e^{-i\pi/4}}{2e^{i\pi/3}} = \frac{3\sqrt{2}}{2}e^{-i(\pi/4+\pi/3)}$$

$$\boxed{= \frac{3\sqrt{2}}{2}e^{-i\,7\pi/12}}$$

5 Applications to Partial Fractions

We want to see how complex numbers can be used to help with some Calculus topics. The first concept is partial fractions. What was the issue with handling irreducible polynomials before?

We have no way to make them zero in value substitution.

→ Need to solve bigger systems of equations.

- But! These polynomials have complex roots!
- Plug in this number and solve out from there.

Method:

1. Find Partial Fraction decomp. like normal, including irreducible quadratics like before.
   → Do not use complex numbers to factor irreducible quadratics.

2. Set up for value substitution.

3. Plug in numbers
   → Linear factors like before
   → Plug in one complex root for each quadratic factor.

4. Solve for constants ← Repeated factors still an issue.

5. Integrate.

Example: Use complex numbers to help compute $\int \frac{12x}{(x+1)(x^2+2x+5)}\,dx$

$$x = \frac{-2 \pm \sqrt{4-20}}{2} = \frac{-2 \pm \sqrt{-16}}{2} = -1 \pm 2i$$

Partial Fractions

$$\frac{12x}{(x+1)(x^2+2x+5)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+2x+5}$$

$$12x = A(x^2+2x+5) + (Bx+C)(x+1)$$

$x=-1$

$$-12 = A(1-2+5) + 0$$

$$-12 = 4A$$

$$A = -3$$

$x=-1+2i$

$$12(-1+2i) = A(0) + (B(-1+2i)+C)(2i)$$

$$-12+24i = 2i(-B+2Bi+C)$$

$$= -2iB + 4Bi^2 + 2iC$$

$$-12 + 24i = -4B + i(2C - 2B)$$

$$-12 = -4B$$

$$B = 3$$

$$24 = 2C - 2B$$

$$30 = 2C$$

$$C = 15$$

$$\int \frac{12x}{(x+1)(x^2+2x+5)}\,dx = \int \frac{-3}{x+1} + \frac{3x+15}{x^2+2x+5}\,dx$$

$$= -3\int \frac{1}{x+1}\,dx + 3\int \frac{x+1}{x^2+2x+5}\,dx$$

$$+ 12\int \frac{1}{\underset{(x+1)^2+4}{x^2+2x+5}}\,dx$$

$$= \boxed{-3\ln|x+1| + \frac{3}{2}\ln|x^2+2x+5| + \underset{6}{\frac{12}{\cancel{2}}}\tan^{-1}\left(\frac{x+1}{2}\right) + C}$$

6 Applications to Power Series

Complex numbers are also useful for interpreting power series and the radius of convergence.

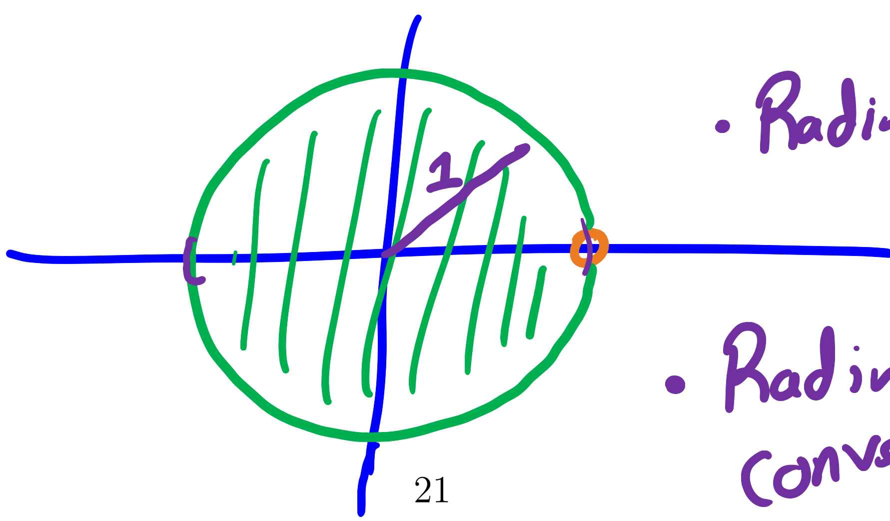
$$\sum_{n=0}^{\infty} a_n x^n$$

- plug in complex numbers instead of just real numbers.

$$\rho = \lim_{n\to\infty} \frac{|a_{n+1} x^{n+1}|}{|a_n x^n|}$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

Converges for $|x| < 1$



- Radius $= 1$
- Radius of convergence.

This can also be tied to power series expansions.

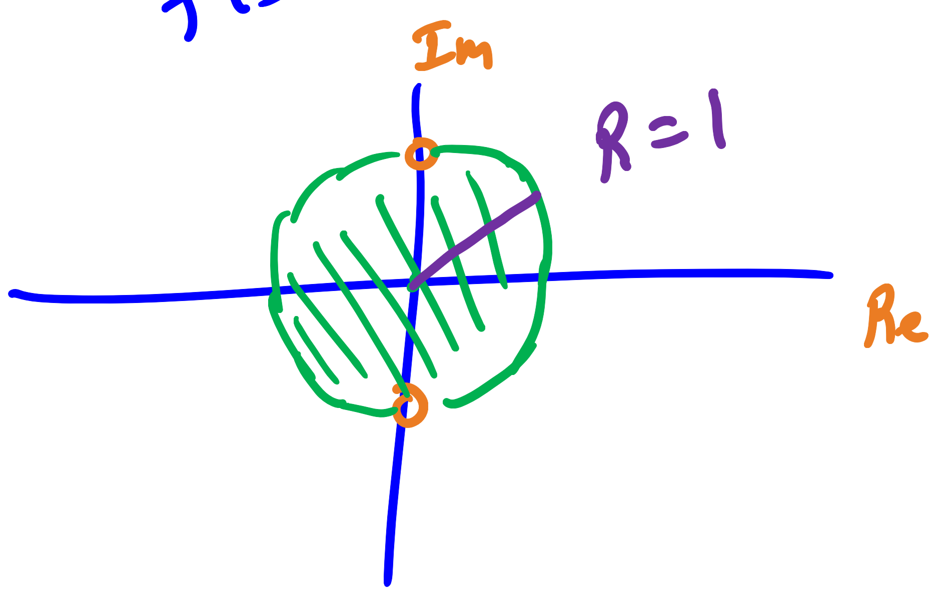
$$f(x) = \frac{1}{1+x^2}$$

Radius of convergence is 1.

→ Defined for all real numbers x.

• But there are complex numbers where it is not defined.

$f(\pm i)$ not defined.



Conclusion:

If there is a place where the function doesn't exist in the **complex plane**, that gives me a point where the series can't converge, and so an upper bound on the radius.

Look for points in the complex plane where the function is not defined

→ Include real values.

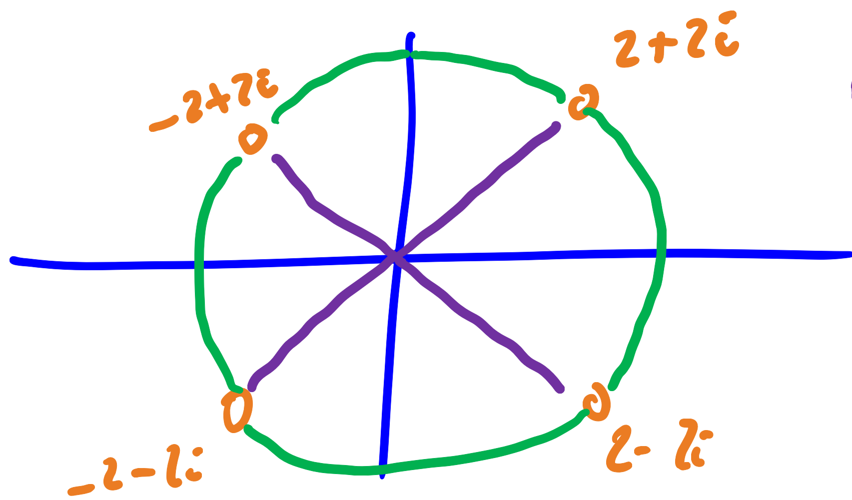
- Expand out from the center until you hit one of those points

→ Radius of Convergence.

Example: Show that the function $f(x) = \frac{x}{x^4+64}$ is undefined at the four complex numbers given by $\pm 2 \pm 2i$. Use this fact to show that the radius of convergence of the power series for $f(x)$ centered at zero is no more than $2\sqrt{2}$. Find the actual power series and validate this.

$$(2 \pm 2i)^2 = 4 \pm 8i + 4i^2 = \pm 8i$$

$$(-2 \pm 2i)^2 = 4 \mp 8i + 4i^2 = \mp 8i$$



$R = \sqrt{4+4} = 2\sqrt{2}$

$$\frac{x}{x^4+64} = \frac{x}{64}\left(\frac{1}{1+\frac{x^4}{64}}\right) = \frac{x}{64}\sum_{n=0}^{\infty}\left(\frac{-x^4}{64}\right)^n$$

$$\boxed{\frac{1}{64}\sum_{n=0}^{\infty}\frac{(-1)^n}{64^n}\cdot x^{4n+1}}$$

$$\rho = \lim_{n\to\infty}\frac{\frac{1}{64^{n+1}}|x|^{4n+5}}{\frac{1}{64^n}|x|^{4n+1}}$$

$|x|^4 < 64$ $\quad |x| < 2\sqrt{2}$

Example: Use complex numbers to help find $\int \frac{13}{\underbrace{(x-2)^2}_{\text{repeated factor}}(x^2+9)}\,dx$

Irreducible Quadratic.

$$\frac{13}{(x-2)^2(x^2+9)} = \frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{Cx+D}{x^2+9}$$

$$13 = A(x-2)(x^2+9) + B(x^2+9) + (Cx+D)(x-2)^2$$

$x=2$

$$13 = 0 + B(4+9) + 0$$

$$B = 1$$

$x=3i$

$$13 = 0 + 0\,(C\cdot 3i + D)(3i-2)^2$$

$$13 = (3iC + D)(3i-2)(3i-2)$$

$$= (3i\,C + D)(9i^2 - 6i - 6i + 4)$$

$$\underbrace{13}=(3ic+D)(-5-12i)$$

$$=-15ic-5D-36i^2C-12iD$$

$$=\underbrace{(36C-5D)}+\underbrace{(-12D-15c)i}$$

$$13=36C-5D$$

$$0=-12D-15C \rightarrow$$

$$15C=-12D$$

$$D=\frac{-5}{4}C$$

$$13=36C-5\left(\frac{-5}{4}C\right)$$

$$13=36C+\frac{25}{4}C=\frac{144+25}{4}C$$

$$=\frac{169}{4}C$$

$$C=4/13 \qquad D=-5/13$$

$$13=A(x-2)(x^2+9)+1(x^2+9)+\left(\frac{4}{13}x-\frac{5}{13}\right)(x-2)^2$$

$x=0$

$$13 = A(-2)(9) + 1(9) + \frac{-5}{13}4$$

$$13 = -18A + 9 - \frac{20}{13}$$

$$A = \frac{1}{18}\left(\underbrace{9-13}_{-4}-\frac{20}{13}\right) = \frac{1}{18}\left(-\frac{52}{13}-\frac{20}{13}\right)$$

$$= \frac{1}{18}\left(\frac{72}{13}\right) = 4/13$$

$$\int \frac{13}{(x-2)^2(x^2+9)}\,dx = \int \frac{4/13}{x-2} + \frac{1}{(x-2)^2} + \frac{4/13\,x}{x^2+9} + \frac{-5/13}{x^2+9}\,dx$$

$$= \boxed{\frac{4}{13}\ln|x-2| - \frac{1}{x-2} + \frac{2}{13}\ln|x^2+9| - \frac{5}{39}\tan^{-1}\left(\frac{x}{3}\right) + C}$$

Example: Use complex numbers to find an upper bound for the radius of convergence of the power series expansion of $\frac{1}{(x^2-4x+5)^2}$ centered at $x = -1$.

- If the function does not exist at a point, the power series can't converge there.

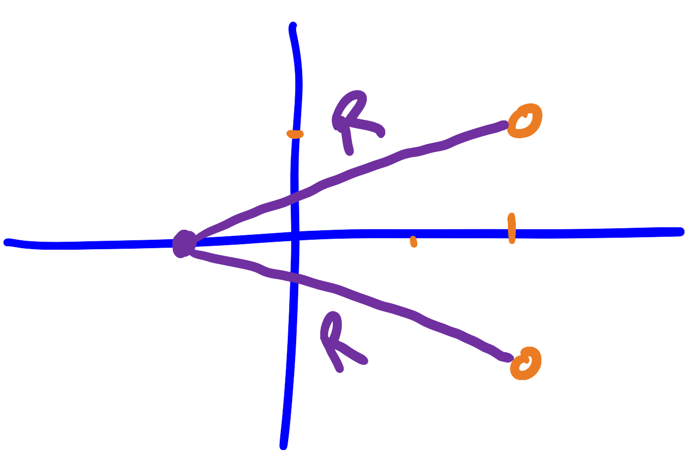
Where is the denominator 0?

$$x^2 - 4x + 5 = 0$$

$$x = \frac{4 \pm \sqrt{16-20}}{2} = \frac{4 \pm \sqrt{-4}}{2}$$

$$= \frac{4 \pm 2i}{2}$$

$$= 2 \pm i$$



$$R = \sqrt{(2 - -1)^2 + (1-0)^2}$$

$$= \sqrt{3^2 + 1^2}$$

$$= \sqrt{10}$$

$$\boxed{\text{Radius is } \le \sqrt{10}}$$