Arc Length and Surface Area

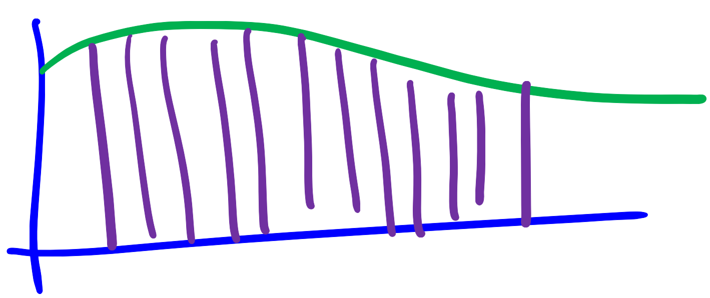
Learning Goals

- Setup and evaluate an x or a y integral for the length of a given curve
- Setup and evaluate an x or a y integral for a surface of revolution about the x or the y-axis

Contents

1 Arc Length

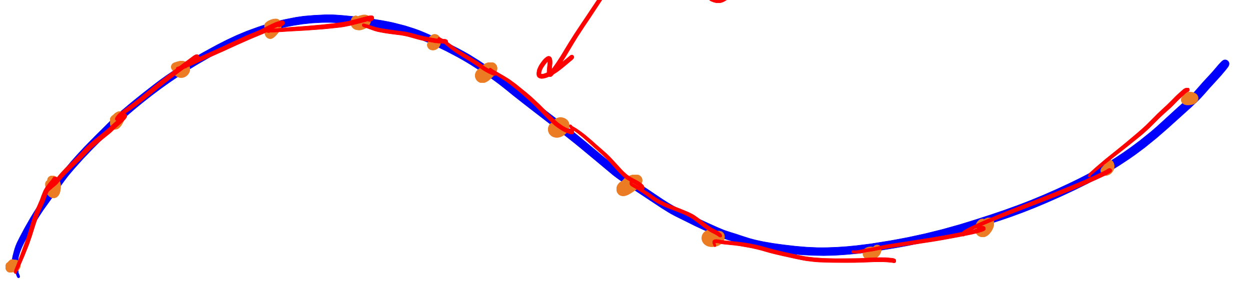
How can we approximate the length of a curve? How did we find the area of a region?

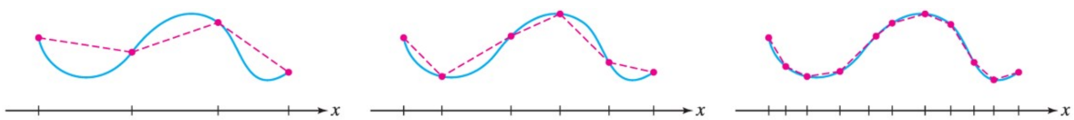


$$\rightsquigarrow \int_a^b f(x)\,dx$$

We want to do the same thing for length: Break it up into little pieces that we know how to compute and add them up, resulting in an integral.

Can find the length of these lines.





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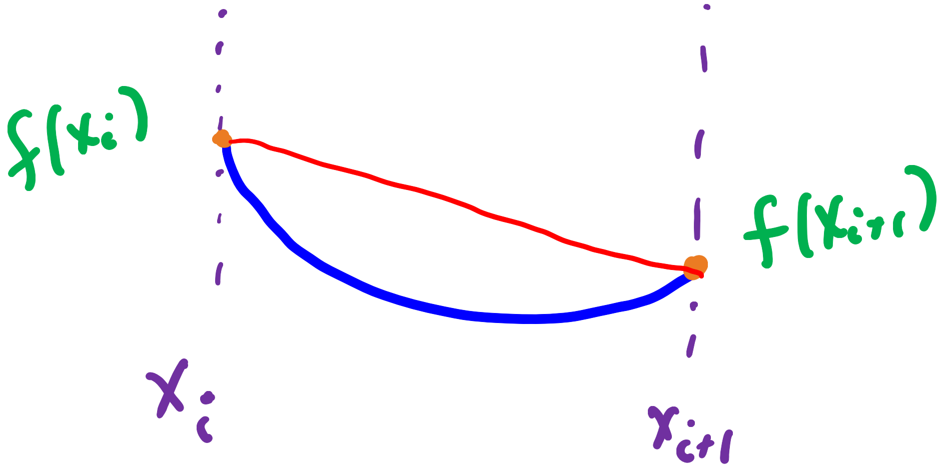
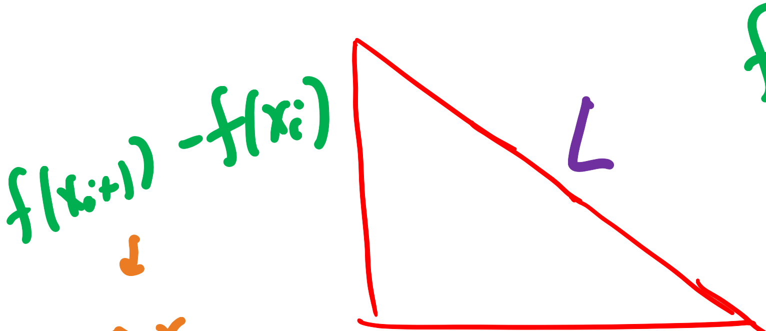
Each of these parts are straight lines. If this is the graph of $f(x)$, then we can find the length of each of these segments.



$$f(x_{i+1}) - f(x_i) = f'(c)\cdot(x_{i+1} - x_i)$$



$f'(c)\,\Delta x$

Δx

$$L = \sqrt{(\Delta x)^2 + f'(c)(\Delta x)^2}$$

$$L = \Delta x\sqrt{1 + f'(c)^2}$$

Formula for Arc Length

Assume that f' exists and is continuous on $[a.b]$. Then the arc length s of $y = f(x)$ over $[a, b]$ is equal to

$$\boxed{s = \int_a^b \sqrt{1 + f'(x)^2}\, dx}$$

$$L = \sqrt{1 + (f'(c))^2}\, \Delta x$$

→ Hard to integrate.

→ May need numerical methods.

Example: Find the length of the curve $y = x^{3/2}$ over the interval $[2, 5]$.

$$S = \int_2^5 \sqrt{1 + (f'(x))^2}\, dx$$

$$f(x) = x^{3/2} \qquad f'(x) = \frac{3}{2} x^{1/2}$$

$$(f'(x))^2 = \frac{9}{4} x$$

$$u = 1 + \frac{9}{4}x$$

$$du = \frac{9}{4}\, dx$$

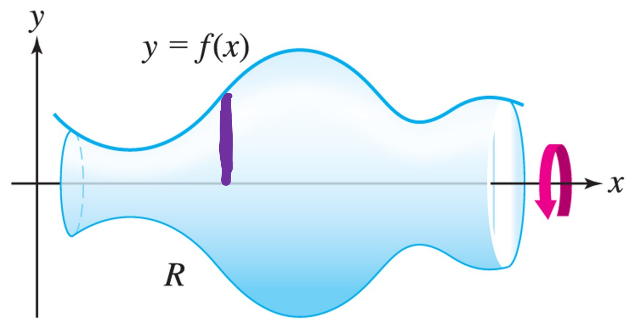
$$S = \int_2^5 \sqrt{1 + \frac{9}{4}x}\, dx$$

$$= \frac{4}{9} \int_{11/2}^{49/4} \sqrt{u}\, du = \frac{4}{9} \cdot \frac{2}{3} u^{3/2} \Big|_{11/2}^{49/4}$$

$$= \boxed{\frac{8}{27}\left(\left(\frac{49}{4}\right)^{3/2} - \left(\frac{11}{2}\right)^{3/2} \right)}$$

2 Surface Area

Another way we can use this idea is in computing the surface area of solids of revolution. Consider a curve $f(x)$ that we want to rotate around the x-axis.



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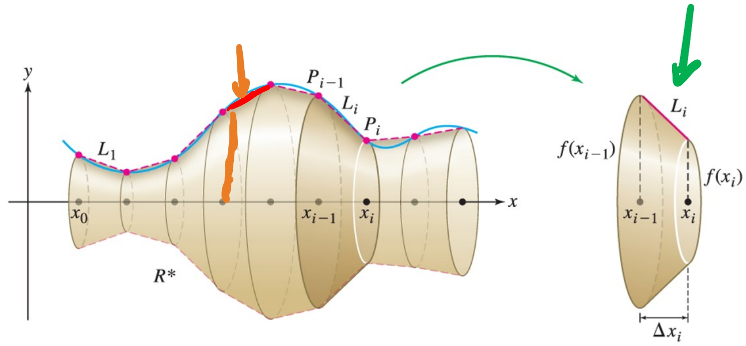
How would we compute the volume?

Washer Method

$$V = \pi \int_a^b (f(x))^2 \, dx$$

How can we use this idea for surface area?



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Need surface area of shells.

$\underline{2\pi r h}$ $\quad r = f(x)$

$h = L = \sqrt{1+f'(x)^2}$

Area of a Surface of Revolution: Assume that $f(x) \geq 0$ and that f' exists and is continuous on the interval $[a, b]$. The surface area S of the surface obtained by rotating the graph of f around the x-axis for $a \leq x \leq b$ is equal to

$$\boxed{S = \int_a^b 2\pi f(x) \sqrt{1+f'(x)^2}\, dx}$$

$2\pi \quad r \quad h$

Example: Find the surface area of the surface of revolution created by revolving the graph of $y = e^{-x}$ over the interval $[2, 4]$ around the x-axis.

$f(x) = e^{-x}$ $\quad f'(x) = -e^{-x}$

$$S = \int_2^4 2\pi e^{-x} \sqrt{1 + e^{-2x}}\, dx$$

$u = e^{-x}$ $\quad du = -e^{-x}\, dx$

$$= 2\pi \int -\sqrt{1+u^2}\, du$$

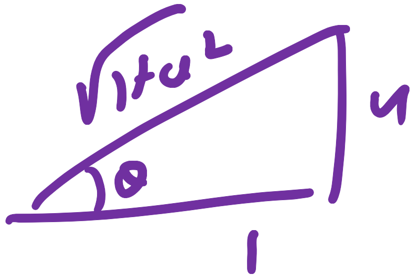
$u = \tan\theta$ $\quad du = \sec^2\theta\, d\theta$

$$= -2\pi \int \sec^3\theta\, d\theta$$

$$= -2\pi \left(\frac{\sec\theta \tan\theta}{2} + \frac{1}{2}\int \sec\theta\, d\theta \right)$$

$$= -\pi \left(\sec\theta \tan\theta + \ln|\sec\theta + \tan\theta| \right) + C$$

$\tan\theta = \frac{u}{1}$



$\sec\theta = \frac{H}{A}$

$$= -\pi\left(u\sqrt{1+u^2} + \ln\left|\sqrt{1+u^2}+u\right|\right)$$

$$= -\pi\left(e^{-x}\sqrt{1+e^{-2x}} + \ln\left|\sqrt{1+e^{-2x}}+e^{-x}\right|\right)$$

→ Plug in 2 and 4 to get actual surface area.

3 More Examples

Example: Find the length of the curve $y = \frac{1}{4}x^2 - \frac{1}{2}\ln x$ over the interval $[1, 4e]$

$$S = \int_1^{4e} \sqrt{1 + (f'(x))^2}\, dx$$

$$f(x) = \frac{1}{4}x^2 - \frac{1}{2}\ln x \qquad f'(x) = \frac{1}{2}x - \frac{1}{2}\frac{1}{x}$$

$$f'(x)^2 = \frac{1}{4}x^2 - \frac{1}{2} + \frac{1}{4}\frac{1}{x^2}$$

$$S = \int_1^{4e} \sqrt{1 + \frac{1}{4}x^2 - \frac{1}{2} + \frac{1}{4}\frac{1}{x^2}}\, dx$$

$$= \int_1^{4e} \sqrt{1/4\, x^2 + 1/2 + 1/4 \frac{1}{x^2}}\, dx$$

$$= \int_1^{4e} \frac{1}{x}\sqrt{1/4\,x^4 + 1/2\,x^2 + 1/4}\;dx$$

$$= \int_1^{4e} \frac{1}{2x}\sqrt{\underbrace{x^4 + 2x^2 + 1}_{(x^2+1)^2}}\;dx$$

$$= \int_1^{4e} \frac{x^2+1}{2x}\,dx = \int_1^{4e} \frac{1}{2}x + \frac{1}{2x}\,dx$$

$$= \frac{x^2}{4} + \frac{1}{2}\ln|x| \Big|_1^{4e}$$

$$= \frac{16e^2}{4} + \frac{1}{2}\underbrace{\ln|4e|}_{\ln 4 + \ln e} - \left(\frac{1}{4} + 0\right)$$

$$= 4e^2 + \frac{1}{2}\ln 4 + 1/2 - 1/4$$

$$= \boxed{4e^2 + \frac{1}{2}\ln 4 + 1/4}$$

Example: Find the arc-length of the curve $y = \frac{1}{x}$ on the interval $[2, 4]$

$$S = \int_2^4 \sqrt{1 + f'(x)^2}\, dx$$

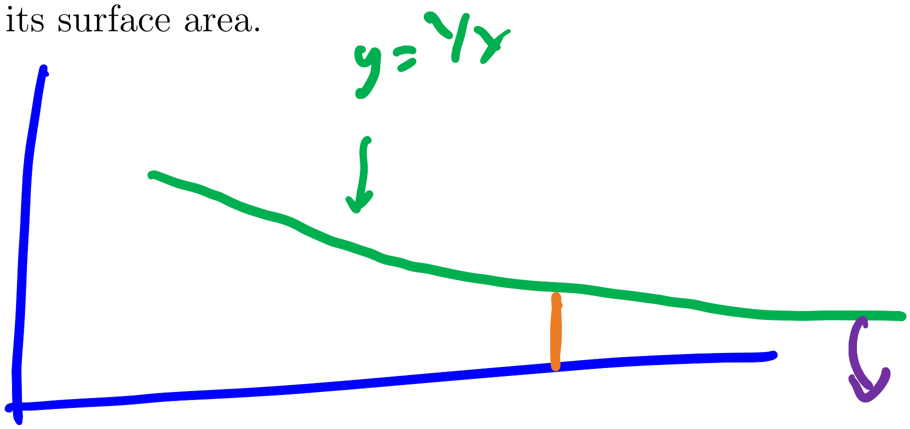
$$f(x) = 1/x \qquad f'(x) = -1/x^2$$

$$S = \int_2^4 \sqrt{1 + 1/x^4}\, dx$$

$$S \sim 2.018$$

4 Gabriel's Horn Example

Example: Gabriel's Horn. Consider the surface generated by rotating the graph of $y = \frac{1}{x}$ on $[1, \infty)$ around the x-axis. Find the volume of this surface of revolution and its surface area.



$$V = \int_1^\infty \pi \left(1/x\right)^2 dx = \pi \int_1^\infty \frac{1}{x^2} dx$$

$$= -\frac{\pi}{x} \Big|_1^\infty = \pi$$

$$SA = \int_1^\infty 2\pi \left(\frac{1}{x}\right) \sqrt{1 + 1/x^4}\, dx \to \text{diverges}$$

$$\frac{1}{x} \cdot \sqrt{1 + 1/x^4} \geq 1/x$$

Infinite Surface Area.

But finite volume.