Numerical Integration

Learning Goals

- Approximate a definite integral using either the midpoint rule, the trapezodial rule or Simpson's rule for a given N
- Compare approximation of a definite integral using either the midpoint rule, the trapezodial rule or Simpson's rule for a given N to the accurate value
- Determine error bounds for either the midpoint rule, the trapezoidal rule or the Simpson's rule
- Determine N for which the error bound for an approximation of a given integral by either the midpoint rule, the trapezoidal rule or the Simpson's rule is given
- Find the constants in an error bound for either the midpoint rule, the trapezoidal rule or the Simpson's rule

Contents

1 Why numerical integration?

Numerical integration is important for several reasons:

1. Only an approximation to the value of an integral is needed.
2. We can't integrate a function by hand.

$\int e^{-x^2}\,dx \rightarrow$ • No "closed-form" expression
• No function for antiderivative

$\int \sin(x^2)\,dx \rightarrow$ Same

$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ → Bell Curve
→ Normal distribution

→ Integrals of this f give us the ability to do statistics.

What is the idea?

We have already thought about integrals as limits of Riemann sums.

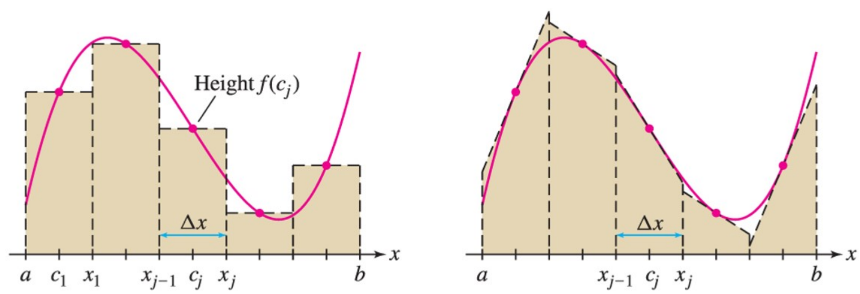
- If I take "enough" rectangles, then the area under the rectangles should be "close" to the area under the curve.

- Also expand beyond rectangles (trapezoids and parabolas) to get different approximations

2 Midpoint and Trapezoid Rule

These are our first two rules for numerical integration. They give us formulas for finding the approximate area, given the function an interval we want to integrate over.

Midpoint Rule



(A) M_N is the sum of the areas of the midpoint rectangles.

(B) M_N is also the sum of the areas of the tangential trapezoids.

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- Take midpoint of each interval, and the function at that point is the height.

M_N = approximation to area using N rectangles, midpoint gives height.

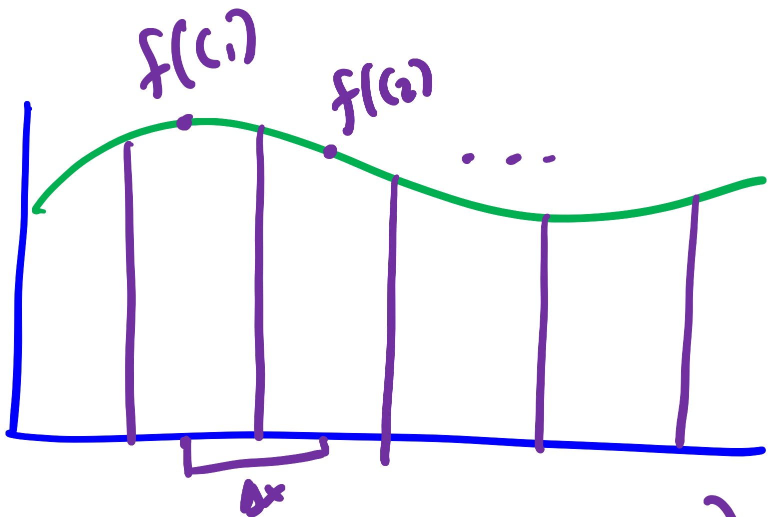
Formula

Interval $[a,b]$

Function $f(x)$.

Approximate

$\int_a^b f(x)\,dx \approx M_N$



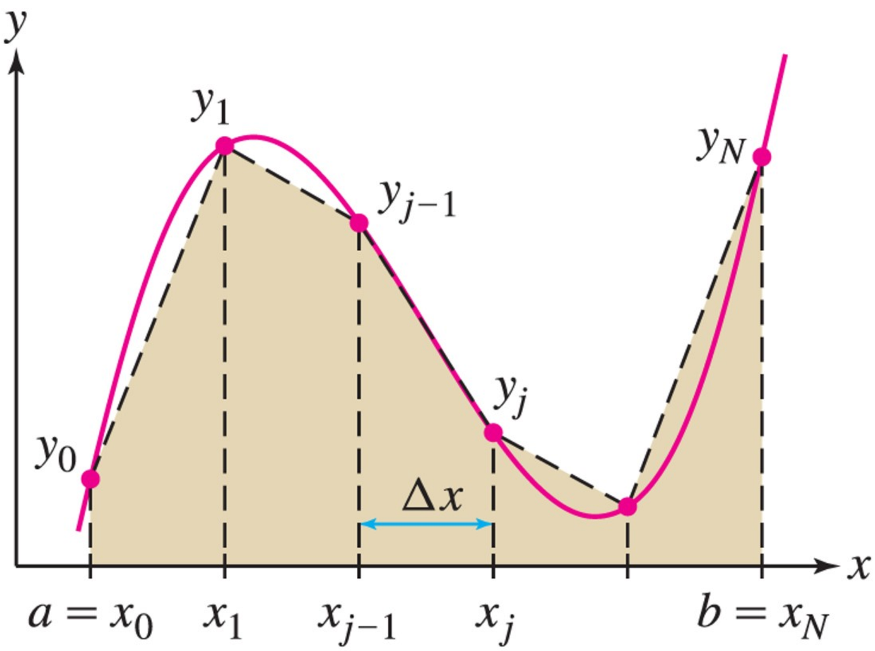
$\Delta x = \dfrac{b-a}{N}$

$M_N = \Delta x f(c_1) + \Delta x(f(c_2)) + \cdots + \Delta x f(c_N)$

$M_N = \Delta x\left(f(c_1) + f(c_2) + \cdots + f(c_n)\right)$

where $\Delta x = \dfrac{b-a}{N}$ $\quad c_i = a + \left(i - \frac{1}{2}\right)\Delta x$

Trapezoid Rule



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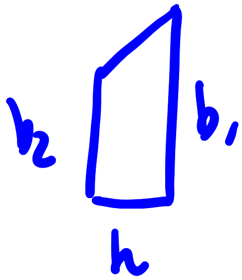
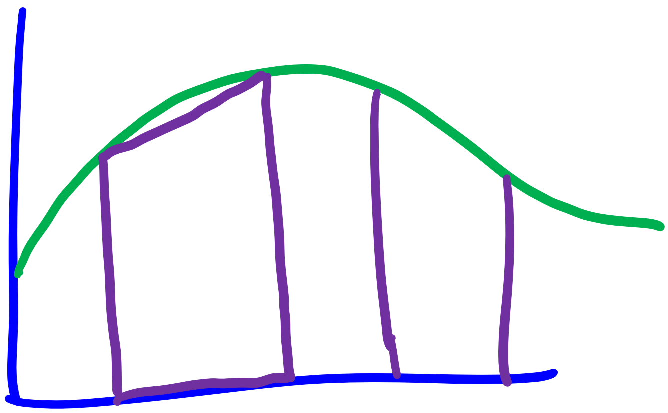
- Connect the two points at each end to make a trapezoid instead of picking the midpoint for a rectangle.
  → Don't need to find midpoints
  → Formula is slightly more complicated

Formula



$$A = h\left(\frac{b_1+b_2}{2}\right) = \frac{h}{2}(b_1+b_2)$$



$h = \Delta x$

$b_1 = f(\text{left end})$

$b_2 = f(\text{right end})$

Take N slices

$x_0 = a \quad x_1 \quad x_2 \quad x_3 \quad \ldots \quad x_N = b$

$$T_N = \frac{\Delta x}{2}\left(f(x_0) + f(x_1)\right) + \frac{\Delta x}{2}\left(f(x_1) + f(x_2)\right) + \frac{\Delta x}{2}\left(f(x_2) + f(x_3)\right) + \ldots \frac{\Delta x}{2}\left(f(x_{N-1}) + f(x_N)\right)$$

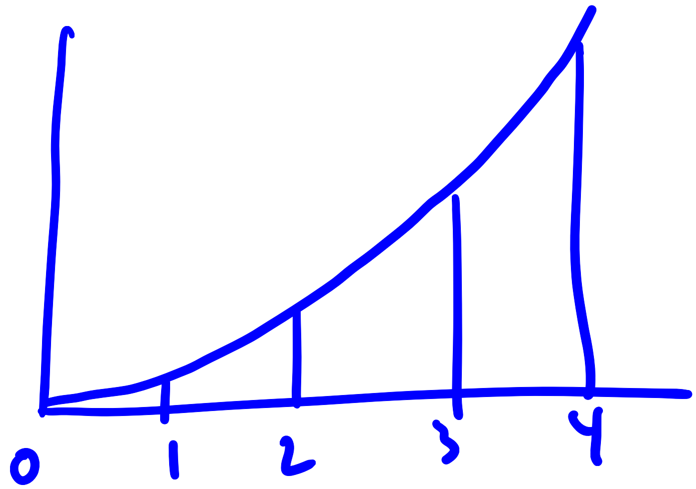
$$T_N = \frac{\Delta x}{2}\left(f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{N-1}) + f(x_N)\right)$$

$$T_N = \frac{\Delta x}{2}\left(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{N-1} + y_N\right)$$

$$\Delta x = \frac{b-a}{N}$$

$$\int_0^4 x^2\,dx = \frac{x^3}{3}\Big|_0^4 = 64/3$$

Example: Compute M_4 and T_4 for the function $f(x) = x^2$ on the interval $[0, 4]$.



$\Delta x = 1$

$$M_4 = \Delta x\left[f(0.5) + f(1.5) + f(2.5) + f(3.5)\right]$$

$$= 1\left[1/4 + 9/4 + 25/4 + 49/4\right]$$

$$= 84/4 = \boxed{21}$$

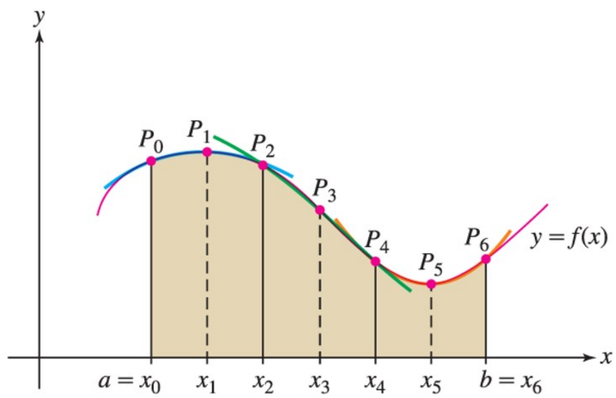
$$T_4 = \frac{\Delta x}{2}\left[y_0 + 2y_1 + 2y_2 + 2y_3 + y_4\right]$$

$$= \frac{1}{2}\left[0 + 2 + 8 + 18 + 16\right]$$

$$= 44/2 = \boxed{22}$$

3 Simpson's Rule

What if I want to do better than straight lines between the points? This thought leads to Simpson's Rule.



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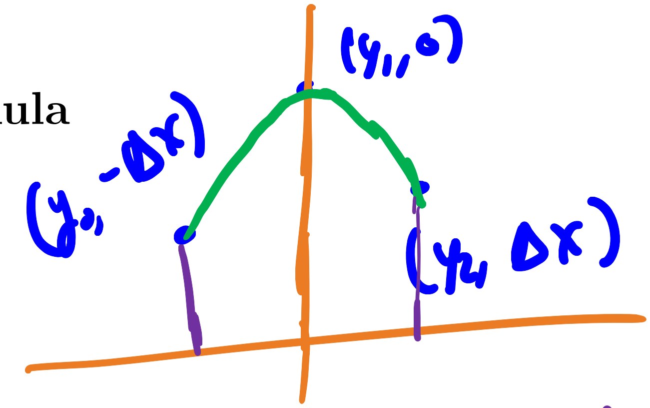
- Trapezoid Rule: Draw line between the endpoints and use that to approximate the function.

- Simpson's Rule: Take three points and draw the parabola through them.

→ Tile this to give the area under the function.

Formula



$$\text{Area} = \frac{\Delta x}{3}\left(y_0 + 4y_1 + y_2\right)$$

$$\text{Area} = \frac{\Delta x}{3}\left(l + 4m + r\right)$$

$$Ax^2 + Bx + C$$

Assume N is even

$$S_N = \frac{\Delta x}{3}\left(y_0 + 4y_1 + y_2\right) + \frac{\Delta x}{3}\left(y_2 + 4y_3 + y_4\right)$$
$$+ \frac{\Delta x}{3}\left(y_4 + 4y_5 + y_6\right) + \cdots$$
$$+ \frac{\Delta x}{3}\left(y_{N-2} + 4y_{N-1} + y_N\right)$$

$$\boxed{S_N = \frac{\Delta x}{3}\left(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + 2y_{N-2} + 4y_{N-1} + y_N\right)}$$

$$\Delta x = \frac{b-a}{N}$$

Example: Compute S_6 for the function $y = x^2$ on $[0, 6]$.

$\Delta x = 1$

$$S_6 = \frac{\Delta x}{3}\left(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + 4y_5 + y_6\right)$$

$$= \frac{1}{3}\left(0 + 4 + 8 + 36 + 32 + 100 + 36\right)$$

$$= \frac{216}{3} = 72$$

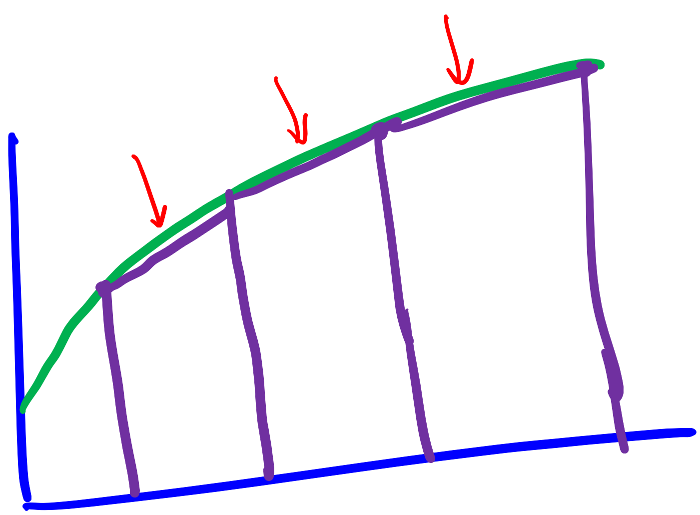
$\to$ Exactly correct!

4 Error Bounds

When doing numerical integration, it's important to know how accurate your approximations are. This is useful in applied settings, when you need to be accurate within a specific amount for calculations to work.

What is an Error Bound?

A way of calculating the "worst case scenario" of how far off my approximate answer is from the actual answer.



If Error $\leq .001$

My answer is correct to 2 decimal places.

Error Bound Formulas

Midpoint + Trapezoid

Assume f'' exists on $[a,b]$, and K_2 is a number so that $K_2 > |f''(x)|$ on $[a,b]$.

Then

$$\boxed{\begin{aligned} \text{error}(M_N) &\le \frac{K_2(b-a)^3}{24N^2} \\ \text{error}(T_N) &\le \frac{K_2(b-a)^3}{12N^2} \end{aligned}}$$

Simpson's Assume that $f^{(4)}(x)$ exists on $[a,b]$ Then and $K_4 > |f^{(4)}(x)|$

$$\boxed{\text{error}(S_N) \le \frac{K_4(b-a)^5}{180N^4}}$$

Example: I want to use numerical integration to compute $\int_0^2 \sqrt{1+x^2}\,dx$. How many terms would I need to guarantee that my approximation was within 0.01 using all three rules?

$K_2 \quad K_4 \qquad (b-a) = 2-0 = 2$

$f(x) = \sqrt{1+x^2}$

$f''(x) = \dfrac{1}{(1+x^2)^{3/2}} \qquad |f''(x)| \leq 1 \quad K_2 = 1$

$f^{(4)}(x) = \dfrac{3(4x^2-1)}{(1+x^2)^{7/2}} \qquad |f^{(4)}(x)| \leq 3 \quad K_4 = 3$

$$\text{error}(M_N) = \frac{K_2(b-a)^3}{24N^2} = .01$$

$$N^2 = \frac{100(8)}{24} = \frac{100}{3}$$

$$N = 5.77$$

Midpoint: 6 rectangles.

$$\text{error}(T_N) = \frac{K_2(b-a)^3}{12N^2} = .01$$

$$N^2 = \frac{200}{3}$$

$$N \approx 8.165$$

Trapezoids: 9 trapezoids

$$\text{error}(S_N) = \frac{K_4(b-a)^5}{180N^4} = .01$$

$$\Rightarrow \frac{3(32)}{180N^4} = .01$$

$$N^4 = \frac{100 \cdot 96}{180} = \frac{160}{3}$$

$$N \approx 7.30$$

Simpsons: 8 points (4 parabolas)