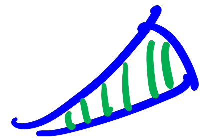
Improper Integrals

Learning Goals

- Identify an integral as an improper integral
- Determine if an improper integral is convergent or divergent by evaluating
- Determine if an improper integral is convergent or divergent using p-integrals
- Determine if an improper integral is convergent or divergent using the comparison test

Contents

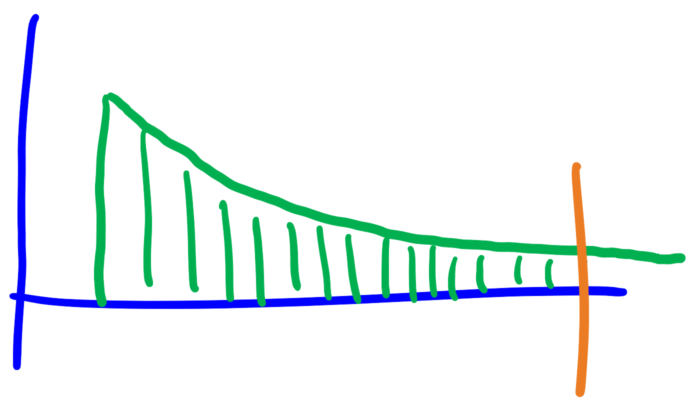


1 Definition

All of the integrals we have dealt with so far represented signed areas/volumes of bounded regions, whether that be area under a single curve, area between two curves, or volumes of revolution. What about unbounded regions? Can we talk about their areas and volumes in the same way? This gives rise to the study of **improper integrals**.

Two types of improper integrals

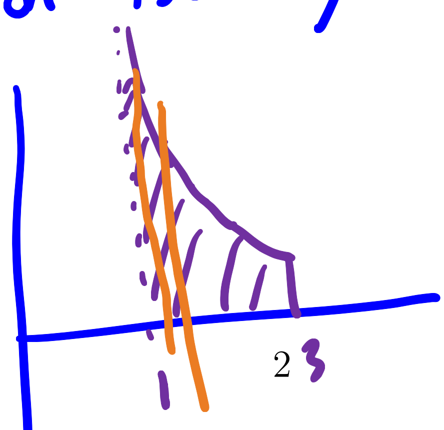
- Bounds of Integration going to infinity



$$\int_2^\infty f(x)\,dx$$

What happens as the cutoff point goes to ∞?

- Function itself goes to infinity



$$\int_1^3 f(x)\,dx$$

What happens when the cutoff point goes to the asymptote?

Make up a cutoff point and then limit it to the problematic point. (∞ or asymptote of function)

If this limit exists, we say the improper integral converges. If not, then the improper integral diverges.

2 Infinite Intervals

How do we analyze $\displaystyle\int_a^\infty f(x)\,dx$?

- Evaluate $\displaystyle\int_a^R f(x)\,dx$

- Look at $\displaystyle\underbrace{\lim_{R\to\infty} \int_a^R f(x)\,dx}$

→ If exists, then converges

→ If does not exist, then diverges
(includes $=\infty$)

Example: Evaluate $\int_2^\infty e^{-4x}\,dx$

1) Evaluate $\int_2^R e^{-4x}\,dx$

$\int_2^R e^{4x}\,dx$

$\frac{1}{4}e^{4x}\Big|_2^R$

$= -\frac{1}{4}e^{-4x}\Big|_2^R$

$\frac{e^{4R}-e^8}{4}$

$= -\frac{1}{4}e^{-4R} - -\frac{1}{4}e^{-8}$

$\downarrow$

∞

$= \frac{e^{-8}-e^{-4R}}{4}$

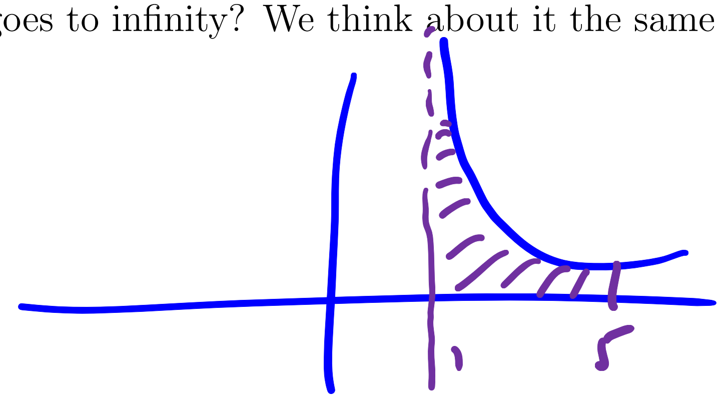
Diverges.

2) $\lim_{R\to\infty} \frac{e^{-8}-e^{-4R}}{4} = \frac{e^{-8}}{4}$ exists

Therefore $\int_2^\infty e^{-4x}\,dx$ converges and equals $\boxed{e^{-8}/4}$

3 Unbounded Integrands

What happens if the integrand goes to infinity? We think about it the same way.



Asymptote or Function is not defined.

1. Figure out where the problematic point is.

$$\int_1^5 f(x)\,dx$$

2. Replace the problematic point by R.
3. Evaluate the integral
4. Limit R to the problematic point from the proper side.

Example: Calculate $\int_1^5 \frac{1}{(x-1)^{2/3}}\,dx$ and $\int_1^5 \frac{1}{(x-1)^{8/3}}\,dx$

- Both functions have an asymptote at $x=1$

$$\int_R^5 \frac{1}{(x-1)^{2/3}}\,dx = 3(x-1)^{1/3}\Big|_R^5$$

$$= 3(4)^{1/3} - 3(R-1)^{1/3}$$

$$\int_1^5 \frac{1}{(x-1)^{2/3}}\,dx = \lim_{R\to 1^+} \int_R^5 \frac{1}{(x-1)^{2/3}}\,dx$$

$$= \lim_{R\to 1^+} 3(4)^{1/3} - \underbrace{3(R-1)^{1/3}}_{0}$$

$$= 3(4)^{1/3}$$ So converges to this!

$$\int_1^5 \frac{1}{(x-1)^{5/3}}\,dx$$

$$\int_R^5 \frac{1}{(x-1)^{5/3}}\,dx = \frac{3}{5}\,\frac{1}{(x-1)^{5/3}}\Bigg|_R^5$$

$$= \frac{3}{5}\left(\frac{1}{4^{5/3}} - \frac{1}{(R-1)^{5/3}}\right)$$

$$\int_1^5 \frac{1}{(x-1)^{5/3}}\,dx = \lim_{R\to 1^+} \frac{3}{5}\left(\frac{1}{4^{5/3}} - \frac{1}{\underbrace{(R-1)}_{0}{}^{5/3}}\right)$$

Limit does not exist

$\int_1^5 \frac{1}{(x-1)^{5/3}}\,dx$ <u>diverges</u>.

4 The p-integral and the Comparison Test

The p-integral is defined as $\displaystyle\int \frac{1}{x^p}\,dx$. What regions would cause this to be an improper integral? When does it converge?

$$\int_0^a \frac{1}{x^p}\,dx \qquad p \neq 1$$

$$\downarrow$$

$$\int_R^a \frac{1}{x^p}\,dx = \frac{1}{1-p}\,x^{1-p}\Big|_R^a$$

$$= \frac{1}{1-p}\left(a^{1-p} - R^{1-p}\right)$$

$$\int_a^\infty \frac{1}{x^p}\,dx$$

$$\int_a^R \frac{1}{x^p}\,dx = \frac{1}{1-p}\left(R^{1-p} - a^{1-p}\right)$$

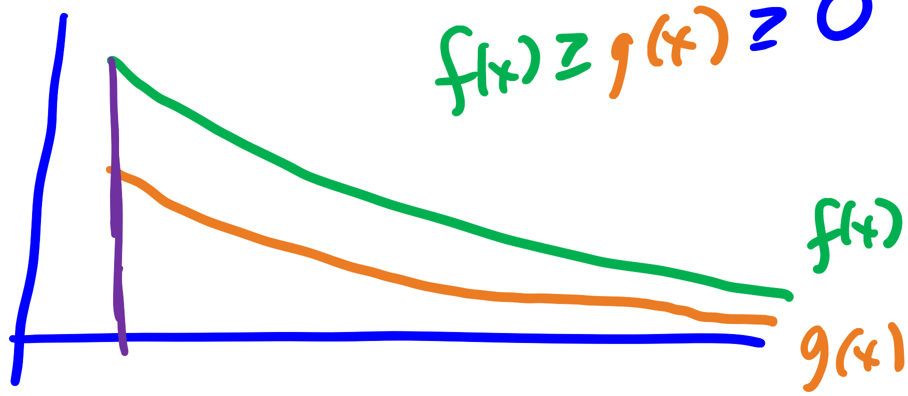
$$\boxed{\int_0^a \frac{1}{x^p}\,dx = \begin{cases} \text{diverges} & p \geq 1 \\ \dfrac{a^{1-p}}{1-p} & p < 1 \end{cases}}$$

$$\boxed{\int_a^\infty \frac{1}{x^p}\,dx = \begin{cases} \dfrac{a^{1-p}}{p-1} & p > 1 \\ \text{diverges} & p \leq 1 \end{cases}}$$

Comparison Test

- Answer convergence without evaluating.

$f(x) \geq g(x) \geq 0$



$\int_1^\infty f(x)\,dx$ does not exist if the region under f contains infinite area.

- Area under g is less than area under f.

If f contains a finite area, then so does g.

If g contains an infinite area, then so does f.

Comparison Test

Let $f(x)$, $g(x)$ be two continuous functions with $f(x) \geq g(x) \geq 0$.

- If $\int_a^\infty f(x)\,dx$ converges, then $\int_a^\infty g(x)\,dx$ also converges.
- If $\int_a^\infty g(x)\,dx$ diverges, then $\int_a^\infty f(x)\,dx$ also diverges.

Example: Does $\int_2^\infty \frac{1}{\sqrt{x}+e^{2x}}\,dx$ converge?

$$e^{2x} \le \sqrt{x} + e^{2x}$$

$$\frac{1}{e^{2x}} \ge \frac{1}{\sqrt{x}+e^{2x}} \ge 0$$

$$\sqrt{x} \le \sqrt{x} + e^{2x}$$

$$\frac{1}{\sqrt{x}} \ge \frac{1}{\sqrt{x}+e^{2x}}$$

diverges

$\int_2^\infty \frac{1}{e^{2x}}\,dx$ converges

$\int_2^\infty e^{-2x}\,dx$

Therefore, by the comp. test

$\boxed{\int_2^\infty \frac{1}{\sqrt{x}+e^{2x}}\,dx \text{ converges}}$

5 Multiple Issues

If there are multiple reasons or places that an integral becomes improper, you have to take care of each one separately.

Example: Evaluate $\displaystyle\int_{-2}^{3} \frac{1}{x^5}\,dx$ $= \dfrac{1}{-4} x^{-4} \Big|_{-2}^{3} = \dfrac{1}{-4}\left((-2)^{-4} - (3)^{-4}\right)$

→ Vertical Asymptote at 0 Makes this an improper integral problem

$$\int_{-2}^{3} \frac{1}{x^5}\,dx = \int_{-2}^{0} \frac{1}{x^5}\,dx + \int_{0}^{3} \frac{1}{x^5}\,dx$$

$$\int_{-2}^{R} \frac{1}{x^5}\,dx = -\frac{1}{4}\left(x^{-4}\right)\Big|_{-2}^{R} \rightarrow \text{diverges}$$

$$= -\frac{1}{4}\Big(\underbrace{R^{-4}}_{\downarrow\,\infty} - (-2)^{-4}\Big)$$

Since $\displaystyle\int_{-2}^{0} \frac{1}{x^5}\,dx$ diverges, so does $\displaystyle\int_{-2}^{3} \frac{1}{x^5}\,dx$

Example: Calculate $\int_{-2}^{1} \frac{1}{x^{2/5}}\,dx$ $= \int_{-2}^{0} \frac{1}{x^{2/5}}\,dx + \int_{0}^{1} \frac{1}{x^{2/5}}\,dx$

- Vertical Asymptote at 0 again.

$$\int_{-2}^{R} \frac{1}{x^{2/5}}\,dx = \frac{5}{3} x^{3/5} \Big|_{-2}^{R} = \frac{5}{3}\left(R^{3/5} - (-2)^{3/5}\right)$$

converges

$$\lim_{R\to 0^-} \frac{5}{3}\left(R^{3/5} - (-2)^{3/5}\right) = -\frac{5}{3}(-2)^{3/5}$$

$$\int_{R}^{1} \frac{1}{x^{2/5}}\,dx = \frac{5}{3} x^{3/5} \Big|_{R}^{1} = \frac{5}{3} - \frac{5}{3} R^{3/5}$$

converges

$$\lim_{R\to 0^+} \frac{5}{3} - \frac{5}{3} R^{3/5} = 5/3$$

so

$$\int_{-2}^{1} \frac{1}{x^{2/5}}\,dx = \boxed{5/3 - \frac{5}{3}(-2)^{3/5}}$$

Example: Evaluate $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx.$ $= \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \int_{0}^{\infty} \frac{1}{1+x^2}\,dx$

- Handle $+\infty$ and $-\infty$ separately

$$\int_R^0 \frac{1}{1+x^2}\,dx = \tan^{-1}(x)\Big|_R^0 = \cancelto{0}{\tan^{-1}(0)} - \tan^{-1}(R)$$

converges

$$\lim_{R\to-\infty} -\tan^{-1}(R) = \pi/2$$

$$\int_0^R \frac{1}{1+x^2}\,dx = \tan^{-1}(x)\Big|_0^R = \tan^{-1}(R) - \cancelto{0}{\tan^{-1}(0)}$$

converges

$$\lim_{R\to\infty} \tan^{-1}(R) = \pi/2$$

So $$\boxed{\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \pi/2 + \pi/2 = \pi}$$