Strategies for Integration

Learning Goals

- Find an appropriate method to evaluate an integral
- Evaluate an integral using a combination of methods of integration
- Evaluate an integral by first applying an algebraic manipulation of the integrand

Contents

1 Summary of Methods

In this section, we put everything from earlier in this chapter together. How do we analyze an integral problem to figure out how we should attack it to get the solution?

What methods do we have?

Cancelling/ Factoring to make things easier.

1. Algebraic simplification
2. Substitution ← u and du
3. Integration by parts ← handling products
4. Special Techniques

→ Trig Integrals
→ Trig Substitution
→ Partial Fractions

→ These have very specific uses.

Algebraic Simplification

- Use normal algebra to make it so you can integrate.
  → Not really an 'integration' technique.

Example: Compute $\int \frac{x^4+3x-5}{x^{3/2}}\,dx$

$$= \int \frac{x^4}{x^{3/2}} + \frac{3x}{x^{3/2}} - \frac{5}{x^{3/2}}\,dx$$

$$= \int x^{5/2} + 3x^{-1/2} - 5x^{-3/2}\,dx$$

$$= \boxed{\frac{2}{7}x^{7/2} + 6x^{1/2} + 10x^{-1/2} + C}$$

2 Substitution Methods

- u-Substitution
  - → If I see a function and its derivative, I can use this to solve the problem.
  - → If we see a composition of functions
- → Rationalizing Substitution

Example: Compute $\int x^7 \sqrt{1+x^4}\, dx$

$u = 1 + x^4 \qquad du = 4x^3\, dx$

$$= \int \frac{1}{4} x^4 \sqrt{u}\; du$$

$$= \frac{1}{4} \int (u-1)\sqrt{u}\; du$$

$$= \frac{1}{4} \int u^{3/2} - u^{1/2}\; du$$

$$= \frac{1}{4}\left(\frac{2}{5} u^{5/2} - \frac{2}{3} u^{3/2}\right) + C$$

$$= \boxed{\frac{1}{10}(1+x^4)^{5/2} - \frac{1}{6}(1+x^4)^{3/2} + C}$$

Example: Compute $\displaystyle\int \frac{x^{1/4}}{x^{1/3} - 16}\, dx$

Rationalizing Substitution.

$u = x^{1/12}$

$u^4 = x^{1/3}$

$x^{1/4} = u^3$

$du = \frac{1}{12} x^{-11/12}\, dx$

$12 x^{11/12}\, du = dx$

$12 u^{11}\, du = dx$

$$= \int \frac{u^3}{u^4 - 16} \cdot 12u^{11}\, du$$

$$= 12 \int \frac{u^{14}}{u^4 - 16}\, du$$

$(u^2+4)(u^2-4)$

$(u^2+4)(u+2)(u-2)$

3 Integration by Parts

$$\int u\,dv = uv - \int v\,du$$

→ Product of functions to integrate.

→ Certain functions that need to be differentiated

→ logs and inverse trig.

$x^n f(x)$, $e^x \cos(x)$, $\ln(x)$, $x^5 \ln(x)$

$dv = dx$

- Diff. log + Inv Trig
- Diff. Polynomials
- Do whatever.

Example: Compute $\int_0^1 \arctan(2x)\, dx$

Integration by parts.

$u = \arctan(2x)$ $\qquad v = x$

$du = \frac{2}{1+4x^2}\, dx$ $\qquad dv = dx$

$$\int_0^1 \arctan(2x)\, dx = x \cdot \arctan(2x)\Big|_0^1 - \int_0^1 \frac{2x}{1+4x^2}\, dx$$

$u = 1+4x^2$

$du = 8x\, dx$

$$= x\arctan(2x) - \frac{1}{4}\ln|1+4x^2|\Big|_0^1$$

$$= \boxed{\arctan(2) - \frac{1}{4}\ln|5|}$$

4 Special Techniques

- Trig Integrals
  - Powers of sine/cosine
  - Powers of sec/tan
  - Different Frequencies
    → Pythagorean Identities + Sub
- Trig Substitution → Reduction
  - Powers of (x^2-a^2), (a^2+x^2), (a^2-x^2)
    → Pick appropriate function
      → Convert back to x.
- Partial Fractions
  → Rational Function
    → Find form of decomposition.
      → Find coefficients
      → Integrate.

Example: Compute $\int \frac{\sqrt{9-4x^2}}{x}\,dx$ $\rightarrow$ Trig substitution

$$\int \frac{\sqrt{9-(2x)^2}}{x}\,dx$$

$2x = 3\sin\theta$

$dx = \frac{3}{2}\cos\theta\,d\theta$

$$= \int \frac{\sqrt{9-9\sin^2\theta}}{\cancel{\frac{3}{2}}\sin\theta}\,\cancel{\frac{3}{2}}\cos\theta\,d\theta$$

$$= \int \frac{3\cos\theta\cdot\cos\theta}{\sin\theta}\,d\theta$$

$$= \int 3\cos\theta\cot\theta\,d\theta$$

$u = \cot\theta$ $\qquad$ $v = \sin\theta$

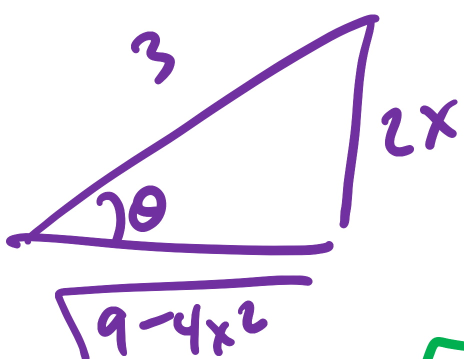
$du = -\csc^2\theta\,d\theta$ $\qquad$ $dv = \cos\theta\,d\theta$

$$= 3\left(\cot\theta \sin\theta + \int \sin\theta\,(+\csc^2\theta)\,d\theta\right)$$

$$= 3\cos\theta + 3\int \csc\theta\,d\theta \quad \frac{\csc\theta + \cot\theta}{\csc\theta + \cot\theta}$$

$$= 3\cos\theta - 3\ln\left|\csc\theta + \cot\theta\right| + C$$

$$2x = 3\sin\theta$$



$$= 3\cdot\frac{\sqrt{9-4x^2}}{3} - 3\ln\left|\frac{3}{2x} + \frac{\sqrt{9-4x^2}}{2x}\right| + C$$

$$= \boxed{\sqrt{9-4x^2} - 3\ln\left|\frac{3+\sqrt{9-4x^2}}{2x}\right| + C}$$