Method of Partial Fractions

Learning Goals

- Find the partial fraction decomposition of a given rational function
- Integrate a rational function by first using long division and then the method of partial fractions
- Integrate a rational function with linear and/or irreducible quadratic factors with multiplicity 1 using the method of partial fractions
- Integrate a rational function with repeated linear and/or irreducible quadratic factors using the method of partial fractions

Contents

1 Partial Fraction Decompositions

In this section, we have one more technique for doing integrals. We'll start by setting this up and then see how it helps with integrals.

What is a Partial Fraction Decomposition?

Can't integrate

- A way to write a complicated rational function as a sum of simpler rational functions.

Can integrate!

Ex $$\frac{x^2-4x+2}{x^3-3x^2+2x} = \frac{1}{x} + \frac{1}{x-1} - \frac{1}{x-2}$$

Can't integrate

Know how to integrate!

Polynomial on top and bottom

What do we know?

If $\frac{p(x)}{q(x)}$ is a rational function with degree of $p(x)$ less than the degree of $q(x)$, and $q(x)$ can be written as

$$q(x) = (x - a_1)(x - a_2) \cdots (x - a_k)$$

→ q completely factors into distinct linear factors.

Then there are constants $A_1, A_2, \cdots, A_k$ so that

$$\frac{p(x)}{q(x)} = \frac{A_1}{(x-a_1)} + \frac{A_2}{(x-a_2)} + \cdots + \frac{A_k}{(x-a_k)}$$

→ Common denominator is $q(x)$.

– $p(x)$ has degree at most $k-1$, which has k coefficients.

Example: Find the Partial Fraction Decomposition for

$$\frac{2x^2 + 5x - 12}{x(x-4)(x+1)}$$

$$\frac{2x^2+5x-12}{x(x-4)(x+1)} = \frac{A}{x} + \frac{B}{x-4} + \frac{C}{x+1}$$

$$= \frac{A(x-4)(x+1) + Bx(x+1) + Cx(x-4)}{x(x-4)(x+1)}$$

$$2x^2+5x-12 = A(x^2-3x-4) + B(x^2+x) + C(x^2-4x)$$

$$2x^2+5x-12 = (A+B+C)x^2 + (-3A+B-4C)x - 4A$$

$$2 = \overset{3}{A}+B+C \quad \rightarrow \quad -1 = B+C$$

$$5 = -3\cancel{A}^{(3)}+B-4C \qquad 14 = B-4C$$

$$-12 = -4A \quad \rightarrow \quad A = 3$$

$$C = -3$$

$$-15 = 5C \qquad B = 2$$

$$\boxed{\frac{2x^2+5x-12}{x(x-4)(x+1)} = \frac{3}{x} + \frac{2}{x-4} - \frac{3}{x+1}}$$

2 Value Substitution and Integrals

For more complicated polynomials, it can be difficult to solve for the necessary coefficients by this method. However, we have another trick we can use.

$$\frac{2x^2+5x-12}{x(x-4)(x+1)} = \frac{A}{x} + \frac{B}{x-4} + \frac{C}{x+1}$$

$$2x^2+5x-12 = A(x-4)(x+1) + Bx(x+1) + Cx(x-4)$$

→ If they are equal at 3 points, they are equal everywhere.

→ Plug in numbers to make this simpler "Value substitution"

$x=0$ $\quad -12 = A(-4)(1) + 0 + 0 = -4A$ $\quad \underline{A=3}$

$x=4$ $\quad 40 = 0 + B4\cdot5 + 0 = 20B$ $\quad \underline{B=2}$

$x=-1$ $\quad -15 = 0 + 0 + C(-1)(-5) = +5C$ $\quad \underline{C=-3}$

Example: Compute $\displaystyle\int \frac{x^2+3}{(x+1)(x+2)(x-4)}\, dx$

$$\frac{x^2+3}{(x+1)(x+2)(x-4)} = \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{x-4}$$

$$x^2+3 = A(x+2)(x-4) + B(x+1)(x-4) + C(x+1)(x+2)$$

$x=-1$ $\quad 4 = A(1)(-5) + 0 + 0 \qquad A = -4/5$

$x=-2$ $\quad 7 = 0 + B(-1)(-6) + 0 \qquad B = 7/6$

$x=4$ $\quad 19 = 0 + 0 + C\cdot 5\cdot 6 \qquad C = 19/30$

$$\int \frac{x^2+3}{(x+1)(x+2)(x-4)}\, dx = \int \frac{-4/5}{x+1} + \frac{7/6}{x+2} + \frac{19/30}{x-4}\, dx$$

$u = x+1$

$$= -\frac{4}{5}\ln|x+1| + \frac{7}{6}\ln(x+2) + \frac{19}{30}\ln|x-4| + C$$

3 Irreducible Quadratics

The method as described previously works when the integrand is a rational function with the following properties:

- The denominator can be completely factored into linear factors
- No linear factor is repeated in this factorization
- The degree of the numerator is less than the degree of the denominator

We will now deal with each of the conditions above, so that we end up with a method that works for all rational functions.

Not all linear factors

Not all polynomials can be completely factored into linear factors.

$x^2+1=0$ $\quad$ x^2+2x+5 $\quad$ "irreducible"

$x^2+1=(x-a_1)(x-a_2)$

Every polynomial can be factored into linear + quadratic terms, so this is the last step we need.

Handling Quadratics

In order to get the right number of coefficients to make these systems work, we need to have both an x term and a constant term on top of the irreducible quadratic.

$$\boxed{\frac{Ax+B}{x^{2}+1}}$$

$$\frac{A}{x^2+2}$$

Every irreducible quadratic that appears adds a term like $\dfrac{Ax+B}{q(x)}$ into the partial fraction decomposition.

$$\frac{Ax}{x^2+1} \rightarrow \text{substitution} \quad + \quad \frac{B}{x^2+1} \rightarrow \text{Inverse tangent}$$

Example: Compute $\displaystyle\int \frac{3x+4}{(x-1)(x^2+9)}\,dx$

$$\frac{3x+4}{(x-1)(x^2+9)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+9}$$

$$3x+4 = A(x^2+9) + (Bx+C)(x-1)$$

$x=1$ $\quad 7 = A(10) + 0 \Rightarrow A = 7/10$

$x=0$ $\quad 4 = A\cdot 9 + (0+C)(-1)$

$$4 = 9\cdot 7/10 - C \Rightarrow C = 23/10$$

$x=2$ $\quad 10 = A(13) + (2B+C)(1)$

$$10 = \frac{7}{10}\cdot 13 + 2B + \frac{23}{10}$$

$$\frac{100}{10} = \frac{91}{10} + \frac{23}{10} + 2B \Rightarrow B = -\frac{7}{10}$$

$$\int \frac{3x+4}{(x-1)(x^2+9)}\,dx = \int \frac{7/10}{x-1} + \frac{-7/10\,x + 23/10}{x^2+9}\,dx$$

$$= \int \frac{7/10}{x-1} + \frac{-7/10\,x}{x^2+9} + \frac{23/10}{x^2+9}\,dx$$

$$\int \frac{7/10}{x-1}\,dx = \frac{7}{10}\ln|x-1| + C$$

$$-7/10 \int \frac{x}{x^2+9}\,dx = \frac{-7}{20}\int \frac{1}{u}\,du = \frac{-7}{20}\ln|x^2+9| + C$$

$u = x^2+9$

$du = 2x\,dx$

$u = \frac{x}{3}$

$du = \frac{1}{3}dx$

$$\frac{23}{10}\int \frac{1}{x^2+9}\,dx = \frac{23}{90}\int \frac{1}{(\frac{x}{3})^2+1}\,dx$$

$$= \frac{23}{30}\int \frac{1}{u^2+1}\,du = \frac{23}{30}\tan^{-1}\left(\frac{x}{3}\right) + C$$

$$\int \frac{3x+4}{(x-1)(x^2+9)}\,dx = \frac{7}{10}\ln|x-1| - \frac{7}{20}\ln|x^2+9| + \frac{23}{30}\tan^{-1}\left(\frac{x}{3}\right) + C$$

4 Repeated Factors

What happens if a factor is repeated in the denominator?

$$\frac{1}{(x-1)^2(x+2)} = \frac{A}{(x-1)^2} + \frac{B}{(x+2)}$$

Needs 2 coeffs on top.

$\underline{(x-1)^2}$ ↳ Quadratic

$$= \frac{Ax + C}{(x-1)^2} + \frac{B}{(x+2)}$$

$$= \frac{\bar{A}}{(x-1)} + \frac{\bar{C}}{(x-1)^2} + \frac{B}{(x+2)}$$

Factor of $(x-a)^k$ in denominator, adds

$$\frac{A_1}{(x-a)} + \frac{A_2}{(x-a)^2} + \cdots + \frac{A_k}{(x-a)^k}$$

to the decomposition

Example: Compute $\int \frac{2x}{(x+1)^2(x-3)}\, dx$

$$\frac{2x}{(x+1)^2(x-3)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x-3}$$

$$2x = A\underline{(x+1)}(x-3) + B(x-3) + C(x+1)^2$$

$x=-1 \quad -2 = 0 + B(-4) + 0 \Rightarrow \underline{B = \frac{1}{2}}$

$x=3 \quad 6 = 0 + 0 + C(4)^2 \Rightarrow C = \frac{6}{16} = 3/8$

$x=0 \quad 0 = A \cdot 1 \cdot (-3) + B(-3) + C$

$3A = -3/2 + 3/8$

$A = -1/2 + 1/8 = -3/8$

$$\frac{2x}{(x+1)^2(x-3)} = \frac{-3/8}{x+1} + \frac{1/2}{(x+1)^2} + \frac{3/8}{x-3}$$

$$\int \frac{2x}{(x+1)^2(x-3)}\,dx = -\frac{3}{8}\int \frac{1}{x+1}\,dx + \frac{1}{2}\int \frac{1}{(x+1)^2}\,dx + 3/8 \int \frac{1}{x-3}\,dx$$

$$= \boxed{-\frac{3}{8}\ln|x+1| + -\frac{1}{2}\,\frac{1}{x+1} + \frac{3}{2}\ln|x-3| + C}$$

5 Long Division

What if the degree on top is higher than the bottom? We can't solve it in the normal way, because we don't have enough information.

$$\frac{x^3+x}{(x+1)(x-1)} = \frac{A}{x+1} + \frac{B}{x-1}$$

$$x^3 + 3x^2 + 4x + 1$$

$$x^3 + x = A(x-1) + B(x+1)$$

$$\begin{array}{r|l} & x \\ \hline x^2-1 & x^3 + \quad x \\ & x^3 - \quad x \\ \hline & \quad 2x \\ \hline \end{array}$$

$$\frac{x^3+x}{x^2-1} = x + \frac{2x}{x^2-1}$$

Integrate normally

Partial Fractions

Example: Compute $\displaystyle\int \frac{x^3+2x+1}{x^2-1}\,dx$

Degree on top is bigger, so long division first.

$$
\begin{array}{r|l}
 & \quad x \\ \hline
x^2-1 & x^3 \qquad + 2x + 1 \\
 & -\,x^3 - x \\ \hline
 & \;\;0 \quad + 3x + 1
\end{array}
$$

$$\frac{x^3+2x+1}{x^2-1} = x + \frac{3x+1}{x^2-1}$$

$$\frac{3x+1}{x^2-1} = \frac{A}{x+1} + \frac{B}{x-1}$$

$$3x+1 = A(x-1) + B(x+1)$$

$x=-1$ $\quad -2 = A(-2) + 0 \Rightarrow A=1$

$x=1$ $\quad 4 = 0 + 2B \Rightarrow B=2$

$$\frac{x^3+2x+1}{x^2-1} = x + \frac{1}{x+1} + \frac{2}{x-1}$$

$$\int \frac{x^3+2x+1}{x^2-1}\,dx = \int x + \frac{1}{x+1} + \frac{2}{x-1}\,dx$$

$$= \boxed{\frac{x^2}{2} + \ln|x+1| + 2\ln|x-1| + C}$$

6 Combining all the Adjustments

Some problems need more than one of these adjustments, and also add in completing a square.

Example: Compute $\int \frac{25}{x(x^2+2x+5)^2}\,dx$

$$\frac{25}{x(x^2+2x+5)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+2x+5} + \frac{Dx+E}{(x^2+2x+5)^2}$$

$$25 = A(x^2+2x+5)^2 + (Bx+C)(x)(x^2+2x+5) + Dx^2 + Ex$$

$x=0$ $\quad 25 = A(25) + 0 + 0 \Rightarrow A=1$

$x=1$ $\quad 25 = 64 + (B+C)(1)(8) + D + E$

$x=-1$ $\quad 25 = 16 + (-B+C)(-1)(4) + D - E$

$x=-2$ $\quad 25 = 25 + (-2B+C)(-2)(5) + 4D - 2E$

$x=-3$ $\quad 25 = 64 + (-3B+C)(-3)(8) + 9D - 3E$

$$-39 = 8B + 8C + D + E$$

$$9 = 4B - 4C + D - E$$

$$0 = \overset{10}{\cancel{20}}B - \overset{5}{\cancel{10}}C + \overset{2}{\cancel{4}}D - \cancel{2}E$$

$$-\underset{13}{\cancel{39}} = \overset{24}{\cancel{72}}B - \overset{8}{\cancel{24}}C + \overset{3}{\cancel{9}}D - \cancel{3}E$$

$$-39 = 18B + 3C + 3D \rightarrow \boxed{-13 = \underline{6}B + C + \underline{D}}$$

$$9 = \underline{-6B} + C - \underline{D}$$

$$-13 = 14B - 3C + D$$

$$-4 = 2C \Rightarrow \underline{C = -2}$$

$$-4 = 8B - 2C \Rightarrow \underline{B = -1}$$

$$-13 = 6(-1) + (-2) + D \quad D = -5$$

$$9 = 4B - 4C + D - E$$

$$E = 4B - 4C + D - 9$$
$$= -4 + 8 - 5 - 9 = \underline{-10}$$

$$\frac{25}{x(x^2+2x+5)^2} = \frac{1}{x} + \frac{-x-2}{x^2+2x+5} + \frac{-5x-10}{(x^2+2x+5)^2}$$

$$u = x^2+2x+5$$
$$du = 2x+2\ dx$$
$$2(x+1)\ dx$$

$$= \frac{1}{x} + \frac{-(x+1)-1}{x^2+2x+5} + \frac{-5(x+1)-5}{(x^2+2x+5)^2}$$

$$\int \frac{25}{x(x^2+2x+5)^2}\,dx = \int \frac{1}{x} - \frac{x+1}{x^2+2x+5} - \frac{1}{x^2+2x+5}$$
$$- 5\frac{x+1}{(x^2+2x+5)^2} - 5\frac{1}{(x^2+2x+5)^2}\ dx$$

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

$$u = x^4 + 2x + 5$$
$$du = 2x + 2\,dx$$

$$-\int \frac{x+1}{x^2+2x+5}\,dx$$

$$= -\frac{1}{2}\int \frac{1}{u}\,du = -\frac{1}{2}\ln|x^2+2x+5| + C$$

$$u = x^2 + 2x + 5$$
$$du = 2x + 2\,dx$$

$$-5\int \frac{x+1}{(x^2+2x+5)^2}\,dx$$

$$-\frac{5}{2}\int \frac{1}{u^2}\,du = \frac{5}{2}\frac{1}{x^2+2x+5} + C$$

$$\int \frac{1}{x^2+2x+5}\,dx \qquad \int \frac{1}{(x^2+2x+5)^2}\,dx$$

$$x^2+2x+5 = (x+1)^2 + 4$$

$$x+1 = 2\tan\theta$$

$$dx = 2\sec^2\theta\,d\theta$$

$$-\int \frac{1}{x^2+2x+5}\,dx = -\int \frac{1}{(x+1)^2+4}\,dx$$

$$= -\int \frac{2\cancel{\sec^2\theta}\,d\theta}{4\cancel{\sec^2\theta}}$$

$$= -\frac{1}{2}\theta + C = -\frac{1}{2}\tan^{-1}\left(\frac{x+1}{2}\right)+C$$

$$-5\int \frac{1}{(x^2+2x+5)^2}\,dx = -5\int \frac{1}{((x+1)^2+4)^2}\,dx$$

$x+1 = 2\tan\theta$

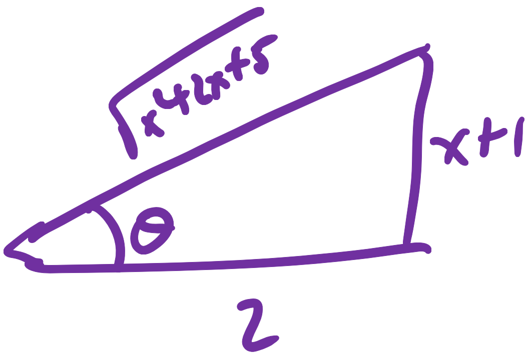
$$= -5\int \frac{2\sec^2\theta\,d\theta}{(4\tan^2\theta+4)^2}$$

$$= -5\int \frac{2\sec^2\theta}{16\sec^4\theta}\,d\theta$$

$$= \frac{-5}{8}\int \cos^2\theta\,d\theta = \frac{-5}{8}\int \frac{1}{2}+\frac{\cos 2\theta}{2}\,d\theta$$

$$= \frac{-5}{8}\left(\frac{1}{2}\theta + \frac{\sin 2\theta}{4}\right) + C$$

$$= \frac{-5}{16}\tan^{-1}\left(\frac{x+1}{2}\right) - \frac{5}{16}\sin\theta\cos\theta + C$$



$$= \frac{-5}{16}\tan^{-1}\left(\frac{x+1}{2}\right) - \frac{5}{16}\cdot\frac{2(x+1)}{x^2+2x+5} + C$$

$$\int \frac{25}{x(x^2+2x+5)^2}\,dx = \ln|x| - \frac{1}{2}\ln|x^2+2x+5| + \frac{5}{2}\cdot\frac{1}{x^2+2x+5} - \frac{1}{2}\tan^{-1}\left(\frac{x+1}{2}\right) - 5/16\tan^{-1}\left(\frac{x+1}{2}\right) - \frac{5}{8}\,\frac{x+1}{x^2+2x+5} + C$$

$$= \ln|x| - \frac{1}{2}\ln|x^2+2x+5| - \frac{13}{16}\tan^{-1}\left(\frac{x+1}{2}\right) - \frac{15-5x}{8(x^2+2x+5)} + C$$