Trigonometric Substitution

Learning Goals

- Evaluate an indefinite and a definite integral using a given trigonometric substitution
- Evaluate an indefinite and a definite integral containing radical using trigonometric substitution
- Evaluate an indefinite and a definite integral containing radical by first completing the square and then trigonometric substitution
- Evaluate an indefinite and a definite integral not containing radicals using trigonometric substitution
- Evaluate an indefinite integral containing a symbolic constant a using trigonometric substitution
- Evaluate an indefinite integral by first substitution and then trigonometric substitution

Contents

1 Pythagorean Identities and Potential Use

Now, we want to use trigonometric functions to solve integrals that do not initially involves these functions. The point is that the relations that trigonometric functions satisfy can help simplify integrals.

Identities

$\sin^2\theta + \cos^2\theta = 1$ $\quad 1 - \sin^2\theta = \cos^2\theta$

$1 + \tan^2\theta = \sec^2\theta$

$\sec^2\theta - 1 = \tan^2\theta$

$\sqrt{1 - \underbrace{\sin^2\theta}_{\text{`u'}}} = \cos\theta$

$du = \cos\theta \, d\theta$

$\sqrt{1 + \tan^2\theta} = \sec\theta$

Goal: - Get rid of square roots
- du term is accounted for by the square root.

Example: Evaluate $\displaystyle\int \frac{1}{(4-x^2)^{5/2}}\,dx$

$$\boxed{x = 2\sin\theta} \qquad dx = 2\cos\theta\, d\theta$$

$$= \int \frac{1}{(4-4\sin^2\theta)^{5/2}} \cdot 2\cos\theta\, d\theta$$

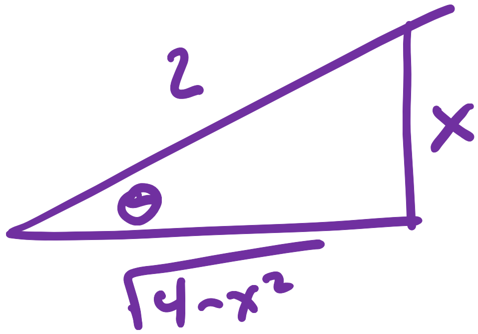
$$= \int \frac{1}{(4\cos^2\theta)^{5/2}} \cdot 2\cos\theta\, d\theta$$

$$= \frac{1}{16}\int \frac{1}{\cos^4(\theta)}\, d\theta = \frac{1}{16}\int \sec^4\theta\, d\theta$$

$$= \frac{1}{16}\int (1+\tan^2(\theta))\sec^2\theta\, d\theta$$

$$= \frac{1}{16}\int 1+u^2\, du = \frac{1}{16}\left(u+\frac{u^3}{3}\right)+C$$

$$= \boxed{\frac{1}{16}\left(\tan\theta+\frac{\tan^3\theta}{3}\right)+C}$$



$$\boxed{= \frac{1}{16}\left(\frac{x}{\sqrt{4-x^2}} + \frac{1}{3}\left(\frac{x}{\sqrt{4-x^2}}\right)^3\right) + C}$$

2 Choosing a Substitution

We have several trigonometric functions to choose from for this method. How do we know which one to use?

→ What does the expression inside the square root look like?

$$1 - \sin^2\theta = \cos^2\theta$$

$$\boxed{a^2 - a^2\sin^2\theta = a^2\cos^2\theta}$$

If I see $\sqrt{a^2 - x^2}$

Plug in $x = a\sin\theta$

$$1 + \tan^2\theta = \sec^2\theta$$

If I see $\sqrt{a^2 + x^2}$

Plug in $x = a\tan\theta$

$$\sec^2\theta - 1 = \tan^2\theta$$

If I see $\sqrt{x^2 - a^2}$

Plug in $x = a\sec\theta$

Example: Compute $\displaystyle\int \frac{dy}{y^2\sqrt{y^2-5}}$.

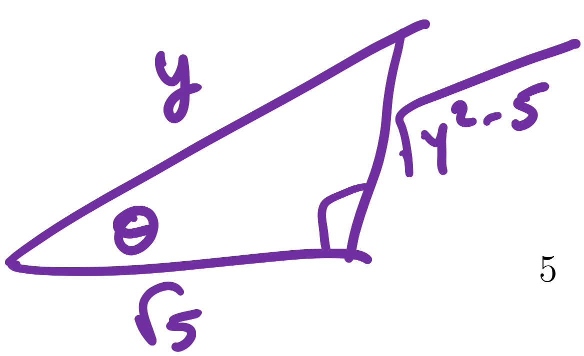
$y = \sqrt{5}\sec\theta$

$dy = \sqrt{5}\sec\theta\tan\theta\, d\theta$

$$= \int \frac{\cancel{\sqrt{5}}\sec\theta\tan\theta\, d\theta}{5\sec^2\theta\sqrt{\underbrace{5\sec^2\theta - 5}_{\cancel{5}\tan^2\theta}}}$$

$$= \frac{1}{5}\int \frac{\cancel{\sec\theta}\,\cancel{\tan\theta}\, d\theta}{\sec^{\cancel{2}}\theta\,\cancel{\tan\theta}}$$

$$= \frac{1}{5}\int \cos\theta\, d\theta = \frac{1}{5}\sin\theta + C$$



$$= \boxed{\frac{1}{5}\frac{\sqrt{y^2-5}}{y} + C}$$

3 Converting Expressions back to x

An important step of this process of using 'trigonometric substitution' is converting the expression back to x after we are done.

Process

- Always using a triangle

- Draw a triangle that relates x to θ using the trigonometric substitution you did at first.
- Solve for the last side of the triangle
- Read the trig functions off the triangle.

☆ Expression that you were trying to simplify away should be the ☆ third side.

Example: Evaluate $\displaystyle\int \frac{dx}{(x^2+49)^{3/2}}$.

$x = 7\tan\theta$

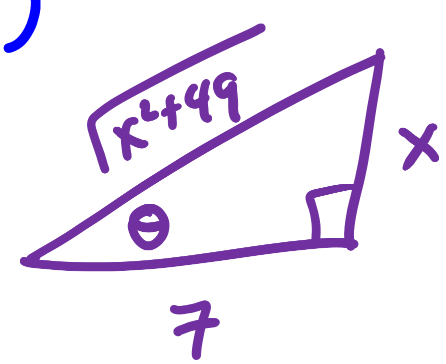
$dx = 7\sec^2\theta\, d\theta$

$$= \int \frac{7\sec^2\theta\, d\theta}{(\underbrace{49\tan^2\theta + 49}_{49\sec^2\theta})^{3/2}}$$

$$= \int \frac{7\sec^2\theta}{49^{3/2}\cdot\sec^3\theta}\, d\theta$$

$$= \frac{1}{49}\int \cos\theta\, d\theta = \frac{1}{49}\sin\theta + C$$

$\tan\theta = \frac{x}{7}$



$$\boxed{= \frac{1}{49}\frac{x}{\sqrt{x^2+49}} + C}$$

$$\cancel{= \frac{1}{49}\sin\left(\tan^{-1}\left(\frac{x}{7}\right)\right) + C}$$

4 Completing the Square

If the expression under the square root is more complicated, we may need one additional step to evaluate the integral.

Any quadratic can be written in the form $\pm(x-c)^2 \pm a^2$ by completing the square

$$\sqrt{x^2+2x+2} = \sqrt{\underbrace{x^2+2x+1}+1}$$

$$= \sqrt{\underbrace{(x+1)^2}_{\tan^2\theta}+1}$$

$x+1 = \tan\theta$

Example: Evaluate $\int \frac{dx}{\sqrt{x^2 + 4x + 1}}$

$\frac{4}{2} = 2$

$2^2 = 4$

$\left(\frac{b}{2}\right)^2 = C$

$$= \int \frac{dx}{\sqrt{x^2+4x+4-3}}$$

$$= \int \frac{dx}{\sqrt{(x+2)^2 - 3}}$$

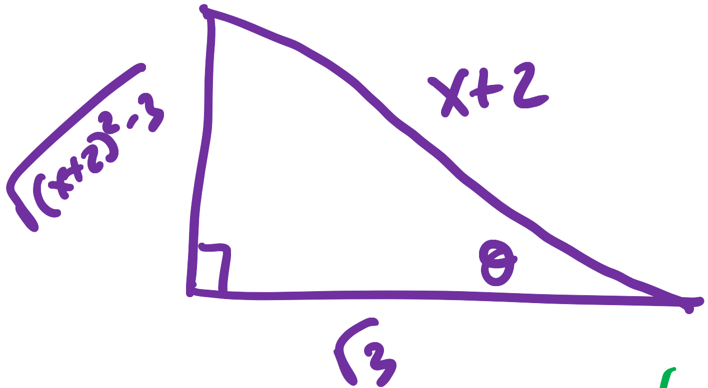
$x+2 = \sqrt{3}\sec\theta$

$dx = \sqrt{3}\sec\theta\tan\theta\, d\theta$

$$= \int \frac{\sqrt{3}\sec\theta\tan\theta\, d\theta}{\sqrt{\underbrace{3\sec^2\theta - 3}_{3\tan^2\theta}}}$$

$$= \int \frac{\cancel{\sqrt{3}}\sec\theta\,\cancel{\tan\theta}\, d\theta}{\cancel{\sqrt{3}}\,\cancel{\tan\theta}}$$

$$= \int \sec\theta\, d\theta = \ln|\sec\theta + \tan\theta| + C$$



$x+2 = \sqrt{3}\sec\theta$

$$\sec\theta = \frac{x+2}{\sqrt{3}} = \frac{H}{A}$$

$$= \boxed{\ln\left| \frac{x+2}{\sqrt{3}} + \frac{\sqrt{(x+2)^2-3}}{\sqrt{3}} \right| + C}$$