Trigonometric Integrals

Learning Goals

- Evaluate an indefinite or definite integral of product of trigonometric functions
- Evaluate an indefinite or definite integral of a unique trigonometric function to a power of a constant, including $\sec^3$
- Evaluate an indefinite or definite integral of product of trigonometric functions with different angles
- Find reduction formulas for integrals containing trigonometric functions to a power of n

Contents

1 Trigonometric Identities

For this section, we need to recall some important trigonometric identities from previous classes. We will need these to help us evaluate integrals.

Pythagorean Identities

$$\sin^2 x + \cos^2 x = 1$$

$$\tan^2 x + 1 = \sec^2 x$$

$$1 + \cot^2 x = \csc^2 x$$

$$\sin^4 x = (1 - \cos^2 x)^2$$

$$= 1 - 2\cos^2 x + \cos^4 x$$

Half-Angle Formulas

$$\sin \frac{x}{2} = \sqrt{\frac{1-\cos x}{2}}$$

$$\cos \frac{x}{2} = \sqrt{\frac{1+\cos x}{2}}$$

$$\boxed{\sin^2 x = \frac{1-\cos 2x}{2}}$$

Product to Sum Formulas

There is another set of identities that are rarely used, but do appear in methods of integration.

$$\sin A \cos B = \frac{1}{2}\sin(A+B) + \frac{1}{2}\sin(A-B)$$
$$\sin A \sin B = \frac{1}{2}\cos(A-B) - \frac{1}{2}\cos(A+B)$$
$$\cos A \cos B = \frac{1}{2}\cos(A+B) - \frac{1}{2}\cos(A-B)$$

Can Integrate!

No formula to integrate

2 Trigonometric Integrals by Substitution

So, what are the integrals that we want to investigate here? The theme of this set is integrals of the form

$$\int \sin^m x \cos^n x \; dx$$

m, n positive integers

If it's just cos(x), easy substitution.

Example: Compute $\int \sin^4 x \underbrace{\cos x \; dx}$

$u = \sin(x)$ $du = \cos(x)\, dx$

$$= \int u^4 \, du = \frac{u^5}{5} + C$$

$$= \boxed{\frac{\sin^5 x}{5} + C}$$

Example: Evaluate $\int \sin^4 x \cos^3 x \; dx$.

$$= \int \sin^4 x \, \cos^2 x \, \underbrace{\cos x \; dx}$$

$\cos^2 x \leftarrow 1 - \sin^2 x$

$u = \sin(x) \qquad du = \cos(x) \; dx$

$$= \int u^4 (1 - u^2) \; du = \int u^4 - u^6 \; du$$

$$= \frac{u^5}{5} - \frac{u^7}{7} + C = \boxed{\frac{\sin^5(x)}{5} - \frac{\sin^7(x)}{7} + C}$$

$$\int \sin^4(x) \cos(x) \; dx \qquad \int \sin^4(x) \cos^3(x) \; dx$$

→ As long as at least one of the powers is odd, I can solve this by substitution

- After pythagorean identities

3 Trigonometric Integrals by Reduction Formulas

There are some choices of m and n in

$$\int \sin^m x \cos^n x \; dx$$

Where m and n are both even.

that we can not handle yet.

Example: Evaluate $\int \sin^6 x \cos^4 x \; dx$

$$= \int \sin^6 x \,(1 - \sin^2 x)^2 \, dx$$

$$= \int \sin^6 x \,(1 - 2\sin^2 x + \sin^4 x) \, dx$$

$$= \int \sin^6 x - 2\sin^8 x + \sin^{10} x \, dx$$

- All even powers.

Let's try the simplest case.

Example: Evaluate $\int \sin^2 x \; dx$

$$\sin\frac{x}{2} = \sqrt{\frac{1-\cos x}{2}}$$

$$\sin^2 x = \frac{1-\cos 2x}{2}$$

$$\int \sin^2 x \, dx = \int \frac{1-\cos 2x}{2} \, dx$$

$$= \int \frac{1}{2} - \frac{\cos 2x}{2} \, dx$$

$$= \boxed{\frac{1}{2}x - \frac{\sin 2x}{4} + C}$$

So, if we could come up with a way to reduce higher powers of sine down to squared, then we could solve the problem. This is where reduction formulas come in to play.

Reduction formula:

$$\boxed{\int \sin^{n}(x)\ dx = -\frac{1}{n}\sin^{n-1}(x)\cos x + \frac{n-1}{n}\int \sin^{n-2} x\ dx}$$

This lets us solve the problem.

Example: Evaluate $\int \sin^6 x \cos^4 x\ dx$

$$= \int \sin^6 x - 2\sin^8 x + \sin^{10} x\ dx$$

$$= \int \sin^{10} x\ dx - 2\int \sin^8 x\ dx + \int \sin^6 x\ dx$$

$$\downarrow$$

$$= -\frac{1}{10}\sin^9(x)\cos(x) + \frac{9}{10}\int \sin^8 x\ dx - 2\int \sin^8 x\ dx + \int \sin^6 x\ dx$$

$$= -\frac{1}{10}\sin^9(x)\cos(x) - \frac{11}{10}\int \sin^8(x)\ dx + \int \sin^6(x)\ dx$$

$$\downarrow$$

$$= -\frac{1}{10}\sin^9(x)\cos(x) - \frac{11}{10}\left[-\frac{1}{8}\sin^7(x)\cos(x) + \frac{7}{8}\int \sin^6 x\ dx\right] + \int \sin^6 x\ dx$$

$$= -\frac{1}{10}\sin^9 x \cos(x) + \frac{11}{80}\sin^7 x \cos(x) - \frac{77}{80}\int \sin^6 x \, dx + \int \sin^6 x \, dx$$

$$= -\frac{1}{10}\sin^9(x)\cos(x) + \frac{11}{80}\sin^7(x)\cos(x) + \frac{3}{80}\int \sin^6(x)\, dx$$

$$= -\frac{1}{10}\sin^9(x)\cos(x) + \frac{11}{80}\sin^7(x)\cos(x) + \frac{3}{80}\left[-\frac{1}{6}\sin^5(x)\cos(x) + \frac{5}{6}\int \sin^4(x)\, dx\right]$$

$$= -\frac{1}{10}\sin^9(x)\cos(x) + \frac{11}{80}\sin^7(x)\cos(x) - \frac{1}{160}\sin^5(x)\cos(x) + \frac{1}{32}\int \sin^4(x)\, dx$$

$$= -\frac{1}{10}\sin^9(x)\cos(x) + \frac{11}{80}\sin^7(x)\cos(x) - \frac{1}{160}\sin^5(x)\cos(x) + \frac{1}{32}\left[-\frac{1}{4}\sin^3(x)\cos(x) + \frac{3}{4}\int \sin^2(x)\, dx\right]$$

$$= -\frac{1}{10}\sin^9(x)\cos(x) + \frac{11}{80}\sin^7(x)\cos(x) - \frac{1}{160}\sin^5(x)\cos(x) - \frac{1}{128}\sin^3(x)\cos(x) + \frac{3}{128}\int \sin^2(x)\, dx$$

$$= -\frac{1}{10}\sin^9(x)\cos(x) + \frac{11}{80}\sin^7(x)\cos(x) - \frac{1}{160}\sin^5(x)\cos(x) - \frac{1}{128}\sin^3(x)\cos x + \frac{3}{128}\left(\frac{1}{2}x - \frac{\sin 2x}{4}\right) + C$$

4 Derivation of Reduction Formulas

All of these reduction formulas are derived from integration by parts.

Example: Find a reduction formula for $\int \sin^n(x)\,dx$

→ Use Integration by Parts

→ Choose u and dv, and dv must be something we can integrate.

$u = \sin^{n-1}(x)$ $\qquad v = -\cos(x)$

$du = (n-1)\sin^{n-2}(x)\cos(x)\,dx$ $\qquad dv = \sin(x)\,dx$

$$\int \sin^n(x)\,dx = -\sin^{n-1}(x)\cos(x) - \int -\cos(x)\,(n-1)\sin^{(n-2)}(x)\cos(x)\,dx$$

$$= -\sin^{n-1}(x)\cos(x) + (n-1)\int \sin^{(n-2)}(x)\cos^2(x)\,dx$$

$$= -\sin^{n-1}(x)\cos(x) + (n-1)\int \sin^{n-2}(x)\left(1-\sin^2(x)\right)dx$$

$$\int \sin^n(x)\,dx = -\sin^{n-1}(x)\cos(x) + (n-1)\int \sin^{n-2}(x)\,dx$$

$$-(n-1)\int \sin^n(x)\,dx$$

$$n\int \sin^n(x)\,dx = -\sin^{n-1}(x)\cos x + (n-1)\int \sin^{n-2}(x)\,dx$$

$$\boxed{\int \sin^n(x)\,dx = -\frac{1}{n}\sin^{n-1}(x)\cos(x) + \frac{n-1}{n}\int \sin^{n-2}(x)\,dx}$$

→ Pick dv to be something we can integrate.

Example: Find a reduction formula for $\int \sec^n(x)\,dx$ $\quad n \geq 2$

$u = \sec^{n-2}(x) \qquad v = \tan(x)$

$du = (n-2)\sec^{n-3}(x)\cdot\sec(x)\tan(x) \qquad dv = \sec^2(x)\,dx$

$$\int \sec^n(x)\,dx = \sec^{n-2}(x)\tan(x) - \int (n-2)\sec^{n-2}(x)\tan^2(x)\,dx$$

$1 + \tan^2(x) = \sec^2(x)$

$$= \sec^{n-2}(x)\tan(x) - (n-2)\int \sec^{n-2}(x)\left(\sec^2(x)-1\right)dx$$

$$\int \sec^n(x)\,dx = \sec^{n-2}(x)\tan(x) - (n-2)\int \sec^n(x)\,dx + (n-2)\int \sec^{n-2}(x)\,dx$$

$$\boxed{\int \sec^n(x)\,dx = \frac{1}{n-1}\sec^{n-2}(x)\tan(x) + \frac{n-2}{n-1}\int \sec^{n-2}(x)\,dx}$$

5 Integrals of Tangent and Secant

A lot of the same techniques that were applied with integrals of the form

$$\int \sin^m x \cos^n x \; dx$$

can also be applied to integrals of the form

$$\int \tan^m x \sec^n x \; dx$$

What are the 'easy' integrals of this form?

If n is even (and at least 2)

Set $du = \sec^2(x)\, dx$ $\quad u = \tan(x)$

If m is odd and $n \geq 1$

$du = \sec(x)\tan(x)\, dx$ $\quad u = \sec(x)$

Use Pythagorean Identities to convert everything to u, then solve.

Hard: n is odd, m is even

Example: Evaluate $\int \sec x \, dx$.

Multiply by "1" $= \dfrac{\sec(x) + \tan(x)}{\sec(x) + \tan(x)}$

$$\int \sec(x)\,dx = \int \frac{\sec^2(x) + \sec(x)\tan(x)}{\sec(x) + \tan(x)}\,dx$$

$u = \sec(x) + \tan(x)$

$du = \sec(x)\tan(x) + \sec^2(x)\,dx$

$$= \int \frac{1}{u}\,du = \ln|u| + C$$

$$= \boxed{\ln\left|\sec(x) + \tan(x)\right| + C}$$

Example: Evaluate $\int \tan^4(x)\sec^4(x)\ dx$

$$= \int \tan^4(x)\sec^2(x)\,\underbrace{\sec^2(x)\,dx}$$

$$= \int \tan^4(x)\left(1+\tan^2(x)\right)\sec^2(x)\,dx$$

$u = \tan(x) \qquad du = \sec^2(x)\,dx$

$$= \int u^4 + u^6\,du$$

$$= \frac{u^5}{5} + \frac{u^7}{7} + C = \boxed{\frac{\tan^5(x)}{5} + \frac{\tan^7(x)}{7} + C}$$

Example: Evaluate $\int \tan^2(x)\sec(x)\ dx$

$\sec^2(x) - 1$

$$\int \sec^3(x)\ dx - \int \sec(x)\ dx$$

$$= \boxed{\frac{1}{2}\sec(x)\tan(x) + \frac{1}{2}\int \sec(x)\ dx} - \int \sec(x)\ dx$$

$$= \frac{1}{2}\sec(x)\tan(x) - \frac{1}{2}\int \sec(x)\ dx$$

$$= \boxed{\frac{1}{2}\sec(x)\tan(x) - \frac{1}{2}\ln\left|\sec(x) + \tan(x)\right| + C}$$

Hard: Odd power of secant

Even power of tangent.

6 Integrals of Products with Different Frequencies

Now, we deal with more integrals of trigonometric functions, but of a different sort.

Example: Evaluate $\int \sin(4x)\cos(3x)\ dx$

$$\sin A \cos B = \frac{1}{2}\underbrace{\sin(A+B)} + \frac{1}{2}\underbrace{\sin(A-B)}$$

$$\int \sin(4x)\cos(3x)\,dx = \int \frac{1}{2}\sin(7x) + \frac{1}{2}\sin(x)\ dx$$

$$= \boxed{\frac{-1}{14}\cos(7x) - \frac{1}{2}\cos(x) + C}$$

→ Convert the product to a sum, then integrate.

The other formulas from the beginning of these notes can be used to compute integrals like

$$1 \qquad \int \sin(5x)\sin(3x)\ dx$$

and

$$2 \qquad \int \cos(2x)\cos(7x)\ dx$$

$$1 = \int \frac{1}{2}\cos(+2x) - \frac{1}{2}\cos(8x)\ dx$$

$$2 = \int \frac{1}{2}\cos(9x) - \frac{1}{2}\cos(+5x)\ dx$$