Integration by Parts

Learning Goals

- Identify when an integral requires integration by parts
- Compute integrals using integration by parts
- Identify which factor of a product should be integrated and which should be differentiated
- Evaluate integrals that require "circular" or repeated integration by parts

Contents

1 Basic Formula

The theme for this next chapter is integration techniques, more tricks to work towards evaluating integrals that aren't just derivatives of functions already known.

The Substitution Method

→ Came from the Chain Rule and trying to work it backwards.

This Section

→ Start with the Product Rule

Product Rule

Recall the product rule for differentiation. Two functions $f(x), g(x)$

$$\frac{d}{dx}\Big(\underbrace{f(x)\,g(x)}\Big) = \underbrace{\left(\frac{d}{dx}f(x)\right)g(x) + f(x)\left(\frac{d}{dx}g(x)\right)}$$

Based on the fact that derivatives and integrals are inverses of each other

$$\int \underbrace{\left(\frac{d}{dx}f(x)\right)g(x)} + f(x)\frac{d}{dx}(g(x))\,dx = f(x)\cdot g(x) + C$$

$$\int f(x)\left(\frac{d}{dx}g(x)\right)dx = f(x)\cdot g(x) - \int g(x)\left(\frac{d}{dx}f(x)\right)dx$$

$$\boxed{\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx}$$

Integration by Parts

$$\boxed{\begin{aligned} \int f(x)\,g'(x)\,dx &= f(x)\,g(x) - \int g(x)\,f'(x)\,dx \\ \int u\,dv &= uv - \int v\,du \end{aligned}}$$

Example: Compute $\int xe^x\, dx$

$u = x \qquad v = e^x$

$du = dx \qquad dv = e^x\, dx$

$$\int u\, dv = uv - \int v\, du$$

$$\int xe^x\, dx = xe^x - \int e^x\, dx$$

$$= \boxed{xe^x - e^x + C}$$

2 Using Integration by Parts

What tricks can be used to analyze an integral to figure out how to apply integration by parts?

- Need to first see it as an integral of a product.
  - → Visualize this product as $u \, dv$.
    - → Need to be able to integrate the dv term.
    - → $\int v \, du$ should be no more complicated than $\int u \, dv$.

More direct Pointers

- → Differentiate logs + Inverse trig.
- → Differentiate polynomials (they go away)
- → Exponential and Trig are fine either way.

Example: Compute $\int x^2 \sin x \; dx$

$u = x^2$ $\qquad v = -\cos(x)$

$du = 2x\,dx$ $\qquad dv = \sin(x)\,dx$

$$\int x^2 \sin(x)\,dx = -\overset{u}{x^2}\overset{v}{\cos(x)} - \int \overset{v}{-\cos(x)} \cdot \overset{du}{2x\,dx}$$

$$= -x^2\cos(x) + \int 2x \cos(x)\,dx$$

$u = 2x$ $\qquad v = \sin(x)$

$du = 2\,dx$ $\qquad dv = \cos(x)\,dx$

$$= -x^2\cos(x) + 2x\sin(x) - \int \sin(x) \cdot 2\,dx$$

$$= \boxed{-x^2\cos(x) + 2x\sin(x) + 2\cos(x) + C}$$

3 Circular Integration by Parts

Some integration by parts problems make it hard to tell there is progress being made.

Example: Compute $\int x^2 \sin x \, dx$

$u = \sin(x)$ $\quad$ $v = x^3/3$

$du = \cos(x)\,dx$ $\quad$ $dv = x^2\,dx$

$$\int x^2 \sin(x)\,dx = \frac{x^3}{3}\sin(x) - \underbrace{\int \frac{x^3}{3}\cos(x)\,dx}$$

→ This integral is more complicated
- Not going to lead to an answer.

Example: Compute $\int e^x \cos x \, dx$

- Clearly a product, so try Integration by Parts.

$$u = e^x \qquad v = \sin(x)$$

$$du = e^x \, dx \qquad dv = \cos(x) \, dx$$

$$\int e^x \cos(x) \, dx = e^x \sin(x) - \int \sin(x) e^x \, dx$$

$$u = e^x \qquad v = -\cos(x)$$

$$du = e^x \, dx \qquad dv = \sin(x) \, dx$$

$$\int e^x \cos(x) \, dx = e^x \sin(x) - 1 \left[e^x (+\cos(x)) - \int -\cos(x) e^x \, dx \right]$$

$$\int e^x \cos(x)\,dx = e^x \sin(x) + e^x \cos(x) - \int e^x \cos(x)\,dx$$

$$2\int e^x \cos(x)\,dx = e^x \sin(x) + e^x \cos(x)$$

$$\int e^x \cos(x) = \boxed{\frac{1}{2}\left(e^x \sin(x) + e^x \cos(x)\right) + C}$$

4 Definite Integration By Parts

Everything discussed so far has been indefinite integrals. What happens if these are definite integrals?

- Idea is the same, just need to deal with the limits at every step.

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$$

$$= u(b)v(b) - u(a)v(a) - \int_a^b v\,du$$

- Can also ignore the limits, solve the problem as an indefinite integral, then plug in at the end.

Example: Compute $\displaystyle\int_2^5 t\ln t\; dt$

- Product, so Integration by Parts.

$u = \ln t$ $\qquad\qquad v = t^2/2$

$du = \frac{1}{t}\,dt$ $\qquad\qquad dv = t\,dt$

$$\int_2^5 t\ln(t)\,dt = \frac{t^2}{2}\ln(t)\Big|_2^5 - \int_2^5 \frac{t^2}{2}\cdot\frac{1}{t}\,dt$$

$$= \frac{t^2}{2}\ln(t)\Big|_2^5 - \frac{1}{2}\int_2^5 t\,dt$$

$$= \frac{t^2}{2}\ln(t)\Big|_2^5 - \frac{t^2}{4}\Big|_2^5$$

$$= \frac{25}{2}\ln(5) - \frac{4}{2}\ln(2) - \left(\frac{25}{4} - \frac{4}{4}\right)$$

$$= \boxed{\frac{25}{2}\ln(5) - 2\ln(2) - \frac{21}{4}}$$

5 Other Integrations by Parts

This technique also allows for computing the integrals of some other functions that could not be done before.

- Differentiating is easier
- Sometimes this is helpful

→ Inverse Trig and Logarithms

$$\int \tan^{-1}(x)\,dx = \tan^{-1}(x)\,x - \int \frac{x\,dx}{1+x^2}$$

$u = \tan^{-1}(x)$ $\quad v = x$

$du = \frac{1}{1+x^2}\,dx$ $\quad dv = dx$

Example: Find $\int \ln x \, dx$

Try integration by Parts

$u = \ln x \qquad v = x$

$du = \frac{1}{x}\,dx \qquad dv = dx$

$$\int \ln x \, dx = x\ln(x) - \int \cancel{x}\,\frac{1}{\cancel{x}}\,dx$$

$$= x\ln(x) - \int 1 \, dx$$

$$= \boxed{x\ln(x) - x + C}$$

$$\left(x\ln(x) - x\right)' = \cancel{x \cdot \frac{1}{x}} + 1 \cdot \ln(x) - \cancel{1} = \underline{\ln(x)}$$