Solids of Revolution - Shell Method

Learning Goals

- Compute volumes of solids of revolution using the shell method in the x-direction
- Compute volumes of solids of revolution using the shell method in the y-direction
- Compute volumes of solids of revolution around lines that are not the axes
- Choose the appropriate method for calculating the volume of a given solid of revolution

Contents

1 Another method 2

2 Multiple Functions and Different Lines 6

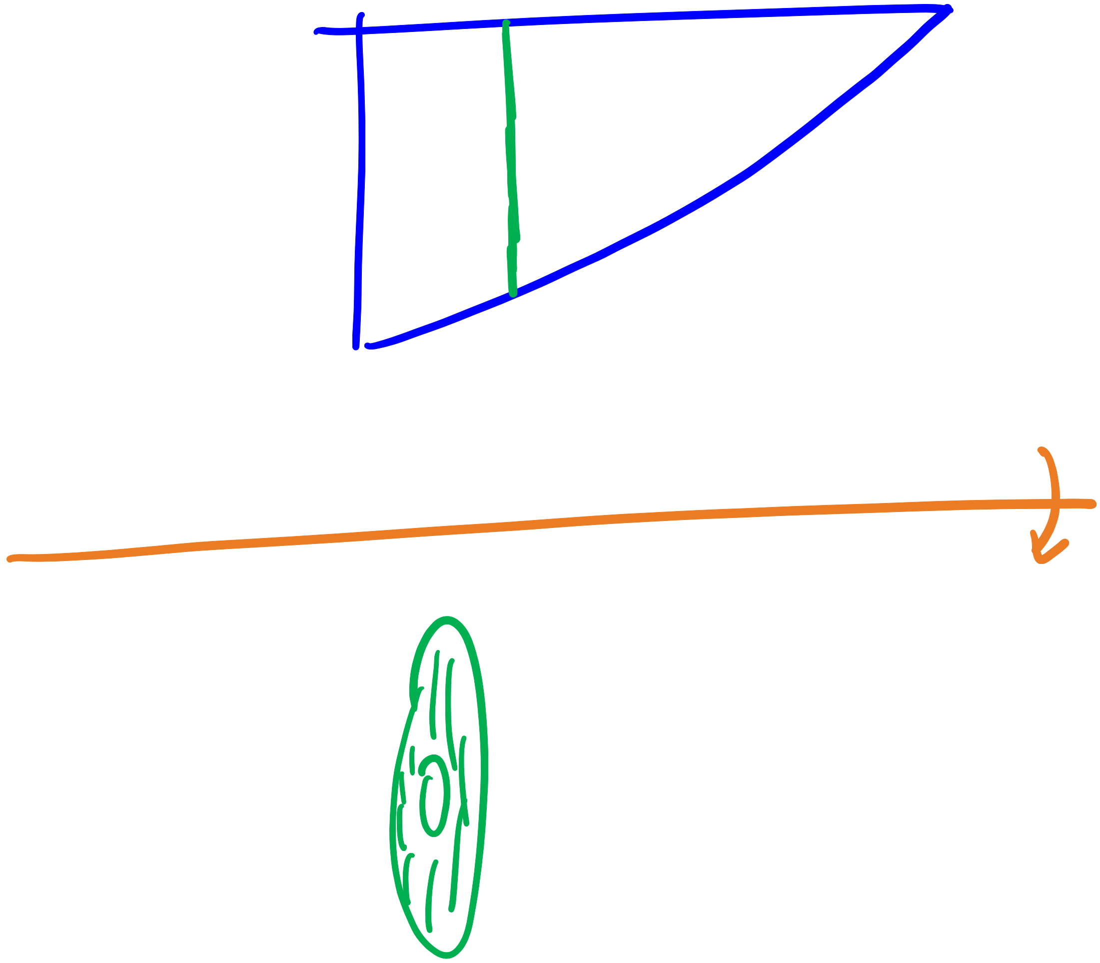
3 Rotating around Horizontal Lines 8

4 Choosing a Method 10

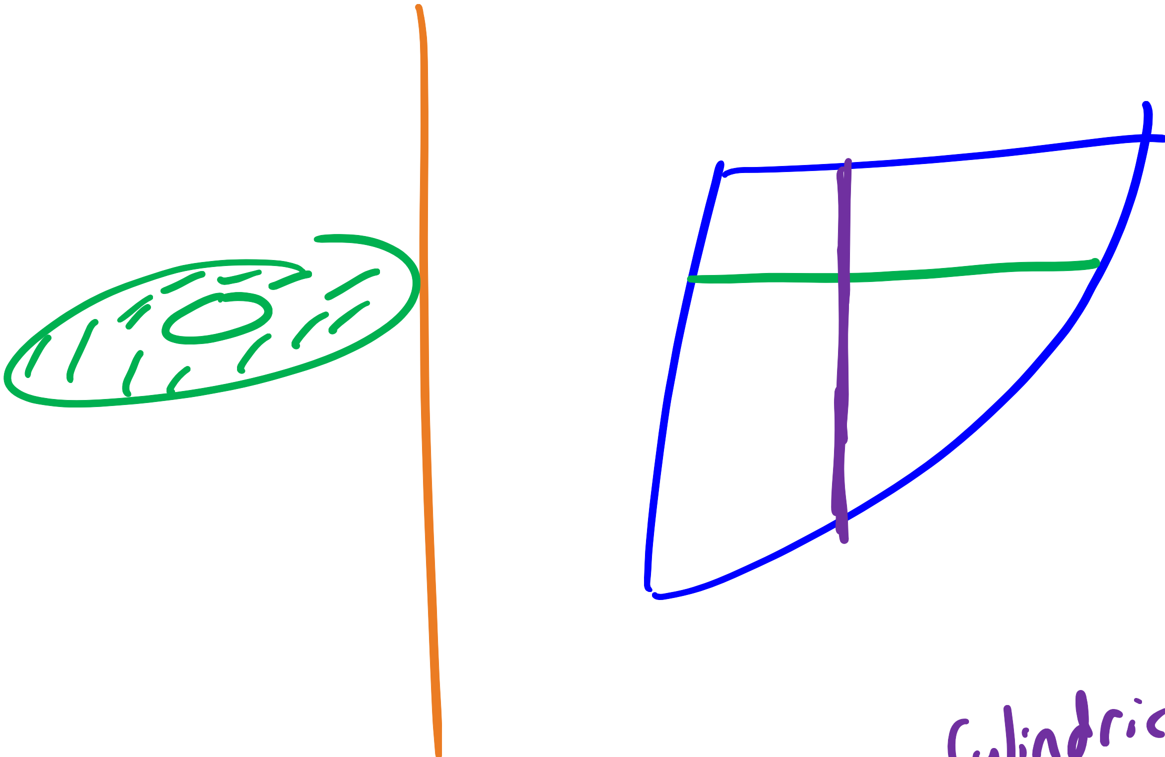
1 Another method

There is one more method that can be used for computing the volume of solids of revolution.

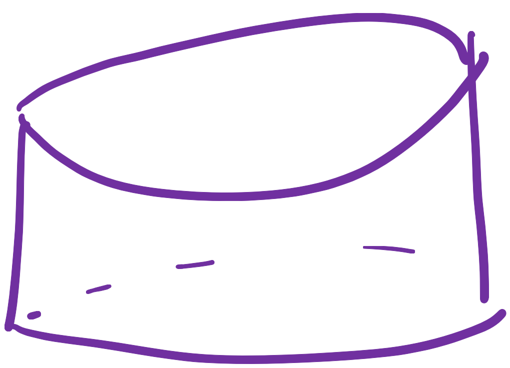
Washer Method



What about the other way?

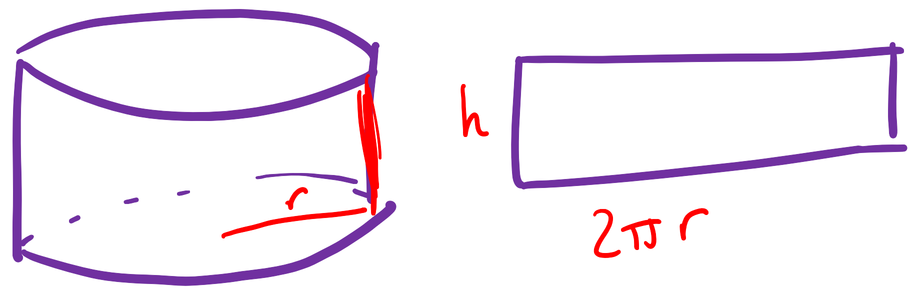


Cylindrical Shells

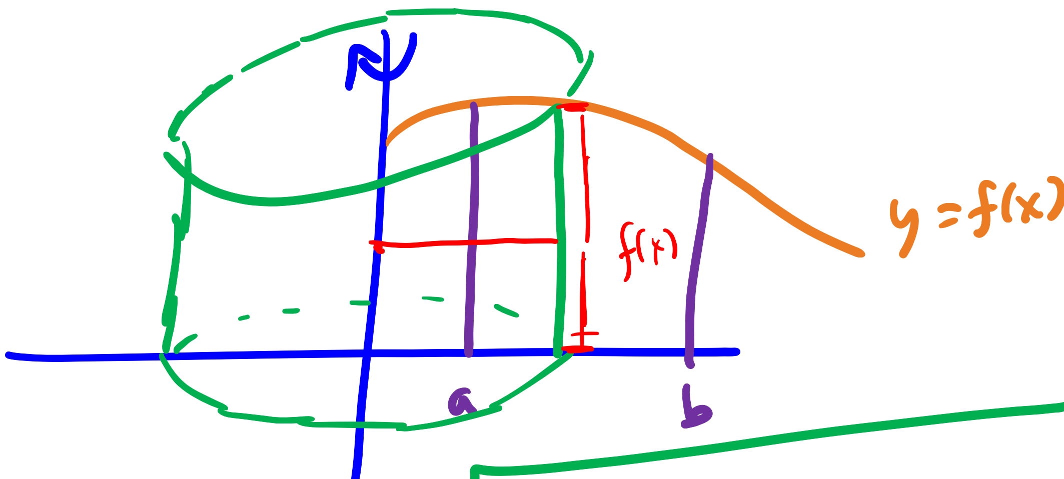


What is the area of this shell?

Shell Method

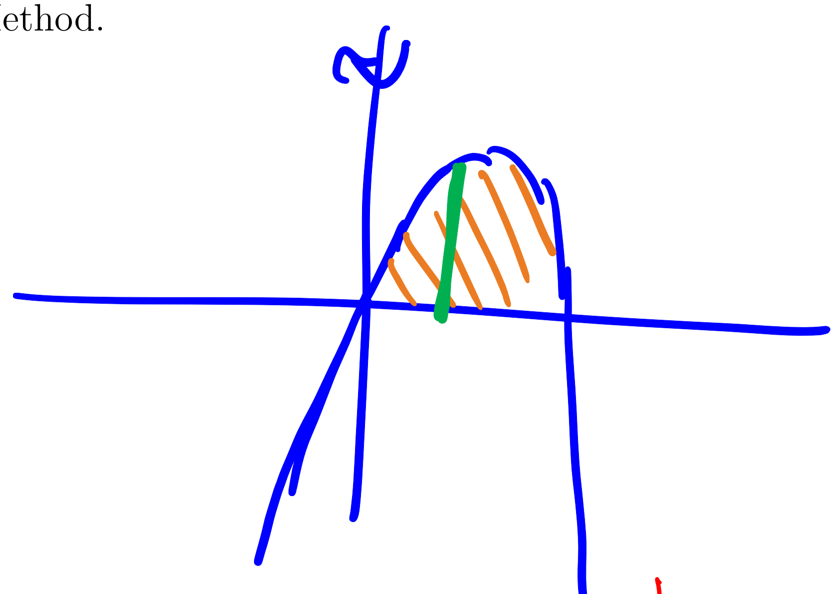


$$A_{shell} = 2\pi r h$$



$$Vol = \int_a^b 2\pi \underset{x}{r}\, \underset{f(x)}{h}\, dx = \boxed{2\pi \int_a^b x f(x)\, dx}$$

Example: Find the volume of the solid obtained by revolving the region between the graph of $y = x(2-x)$ and the x axis around the y-axis using the Shell Method.



$$\text{Vol} = \int_0^2 2\pi \cdot \overset{r}{x} \cdot \overset{h}{x(2-x)}\, dx$$

$$= 2\pi \int_0^2 2x^2 - x^3\, dx$$

$$= 2\pi \left(\frac{2}{3}x^3 - \frac{x^4}{4} \right) \Big|_0^2$$

$$= 2\pi \left(\frac{2}{3} \cdot 8 - \frac{16}{4} \right) = \frac{8}{3}\pi$$

2 Multiple Functions and Different Lines

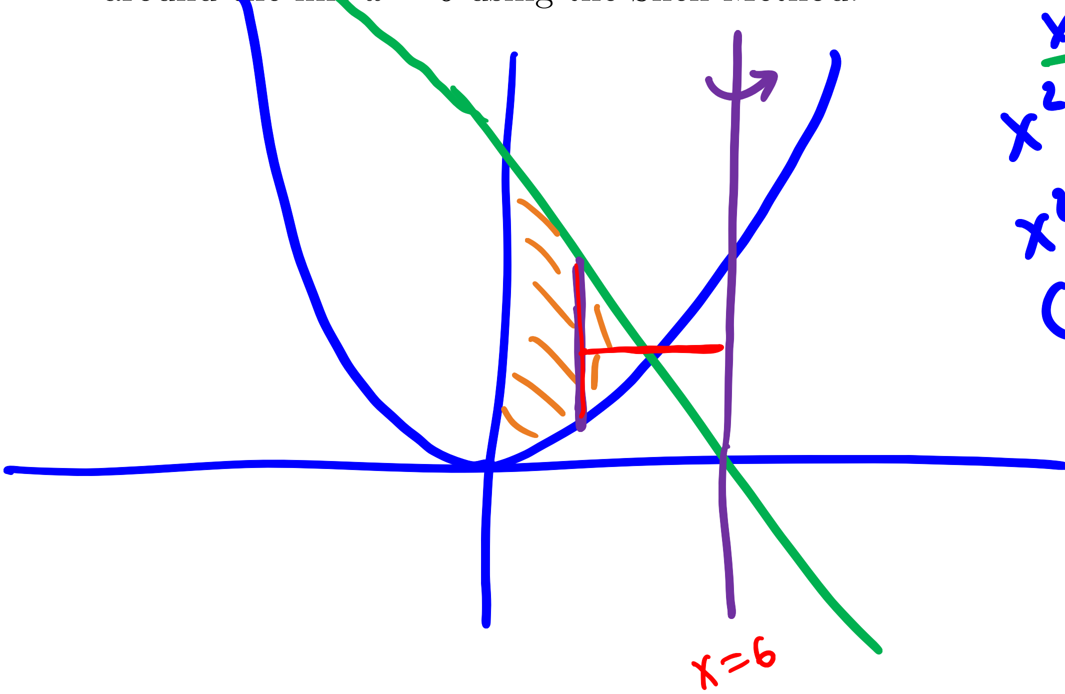
The shell method can also be applied to setups where multiple functions or different rotation axes are involved.

$$\text{Vol} = 2\pi \int_a^b r \cdot h \, dx$$

Need to figure out what radius and height are, along with bounds of integration.

→ Then just plug in and use the formula.

Example: Find the volume of the solid of revolution obtained by revolving the region between the y-axis, the graph $y = x^2$, and the graph of $y = 6 - x$ around the line $x = 6$ using the Shell Method.



$$x = 0$$
$$x^2 = 6 - x$$
$$x^2 + x - 6 = 0$$
$$(x+3)(x-2) = 0$$
$$x = 2$$

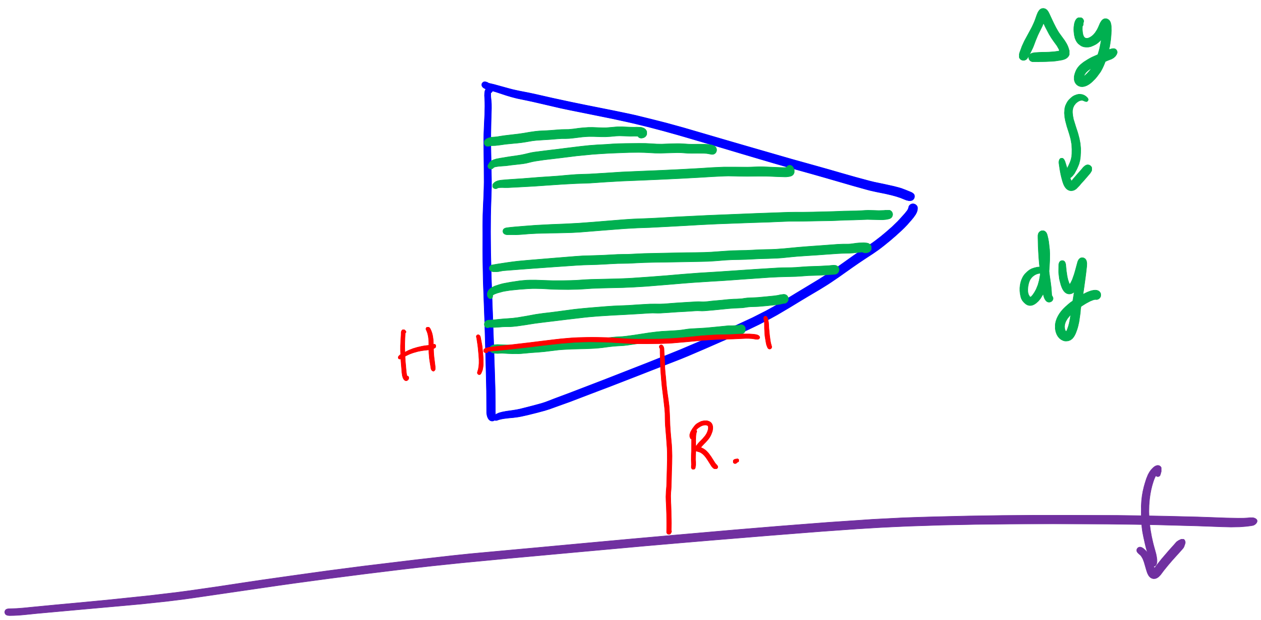
Radius $\rightarrow$ $6 - x$

Height $\rightarrow$ $(6 - x) - x^2 = 6 - x - x^2$

$$\text{Vol} = 2\pi \int_0^2 (6-x)(6-x-x^2)\,dx$$
$$= 2\pi \int_0^2 36 - 6x - 6x^2 - 6x + x^2 + x^3\,dx$$
$$= 2\pi \int_0^2 36 - 12x - 5x^2 + x^3\,dx$$
$$= 2\pi \left(36x - 6x^2 - \frac{5}{3}x^3 + \frac{x^4}{4}\right)\Big|_0^2 = \frac{232\pi}{3}$$

3 Rotating around Horizontal Lines

The Shell Method also works for rotation around horizontal lines. Based on the way the shells are being added up, this requires an integral in y, as opposed to one in x.

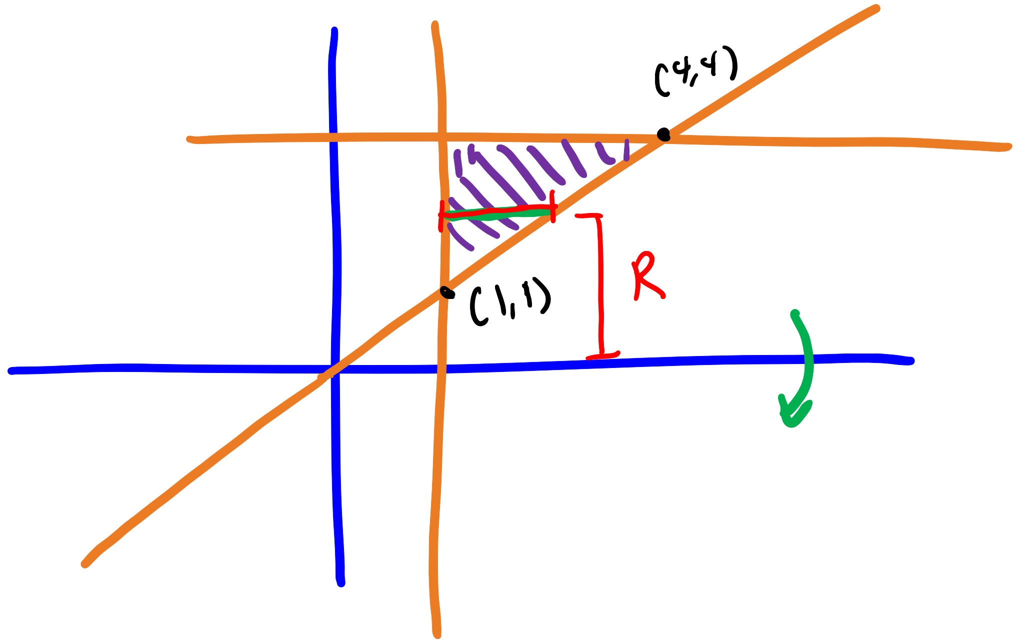


→ Define everything in terms of y

→ Find radius and height in terms of y

Same formula: $2\pi \int_c^d r(y)\,h(y)\,dy$

Example: Find the volume of the solid of revolution obtained by rotating the region between $x = 1$, $y = 4$ and $y = x$ around the x axis using the Shell Method.



Bounds of Integration: $y=1$ to $y=4$

Radius: For a given value of y $\quad r(y) = y$

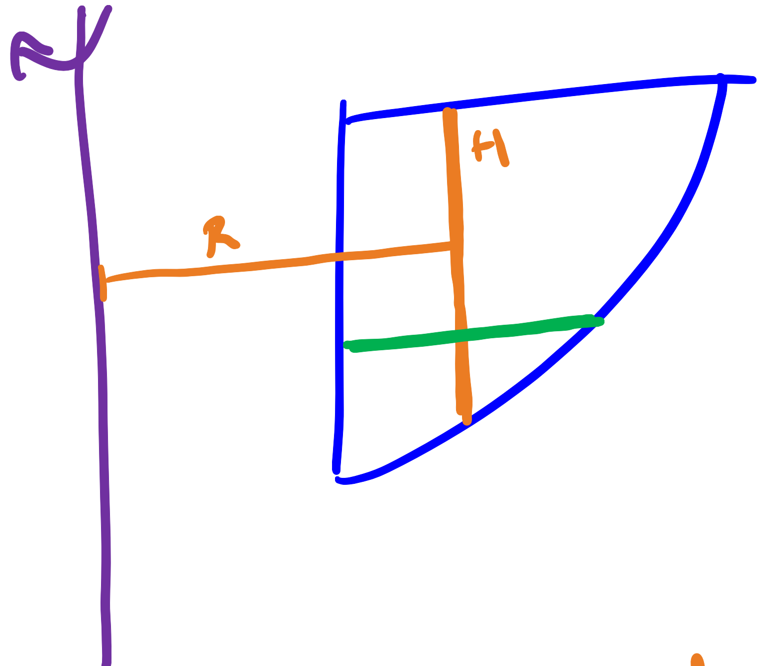
Height: $x_{right} - x_{left} = y - 1$

$$\text{Vol} = 2\pi \int_1^4 y(y-1)\,dy = 2\pi \int_1^4 y^2 - y\,dy$$

$$= 2\pi \left(\frac{y^3}{3} - \frac{y^2}{2} \right)\Big|_1^4 = 27\pi$$

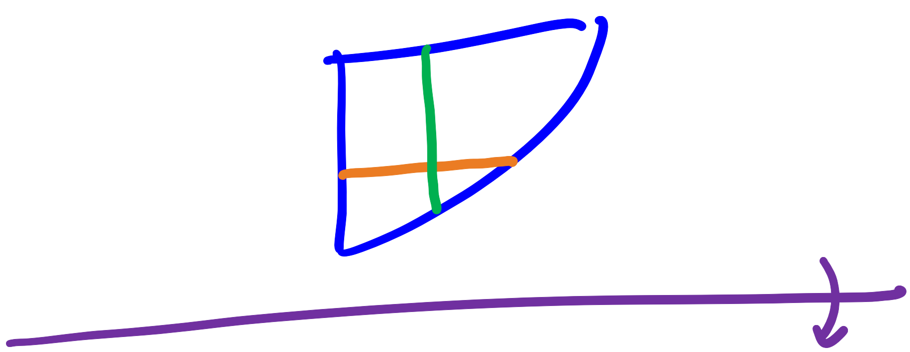
4 Choosing a Method

There are two ways to compute volumes of solids of revolution. Which one is better depends on the situation.



Shells → Find Radius and Height (dx)

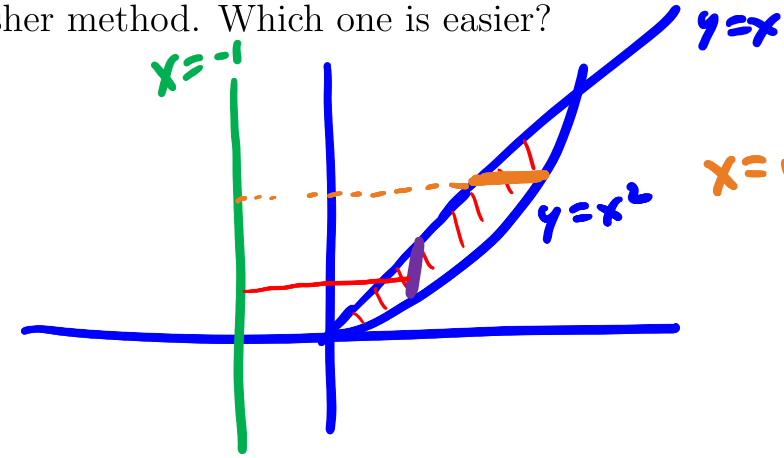
Washers → Find inner and outer radius (dy)



Shells → dy

Washers → dx

Example: Find the volume of the solid of revolution obtained by revolving the region between $y = x$ and $y = x^2$ around the line $x = -1$ by both the shell and washer method. Which one is easier?



$\rightarrow x = y$

$x = \sqrt{y}$

$y+1$ Inner

$\sqrt{y} + 1$

Shells (dx) Radius: $1+x$ Height: $x - x^2$

Bounds: 0, 1

$$\text{Vol: } 2\pi \int_0^1 (1+x)(x-x^2)\,dx$$

$$= 2\pi \int_0^1 x - x^2 + x^2 - x^3\,dx$$

$$= 2\pi \int_0^1 x - x^3\,dx = 2\pi\left(\frac{x^2}{2} - \frac{x^4}{4}\right)\Big|_0^1$$

$$= \boxed{\pi/2}$$

Washers (dy) Inner : $y+1$

Outer : $\sqrt{y}+1$

Bounds: 0, 1

$$\text{Vol} = \pi \int_0^1 (\sqrt{y}+1)^2 - (y+1)^2 \, dy$$

$$= \pi \int_0^1 y + 2\sqrt{y} + \cancel{1} - (y^2 + 2y \cancel{+1})\, dy$$

$$= \pi \int_0^1 -y^2 - y + 2\sqrt{y} \; dy$$

$$= \pi \left(\frac{-y^3}{3} - \frac{y^2}{2} + 2 \cdot \frac{2}{3} y^{3/2} \right) \Big|_0^1$$

$$= \pi \left(-\frac{1}{3} - \frac{1}{2} + \frac{4}{3} \right) = \boxed{\pi/2}$$