Solids of Revolution - Washer Method

Learning Goals

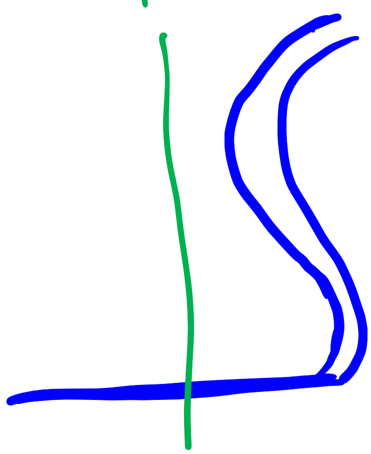
- Determine if something is a solid of revolution
- Compute volumes of solids of revolution using the washer method in the x-direction
- Compute volumes of solids of revolution using the washer method in the y-direction
- Compute volumes of solids of revolution around lines that are not the axes

Contents

1 Solids of Revolution

What is a solid of revolution?

- Anything with a rotational axis of symmetry.



- Form these by rotating a two-dimensional region around an axis in 3D.

Computing Volume

How can the volume of these solids be found?

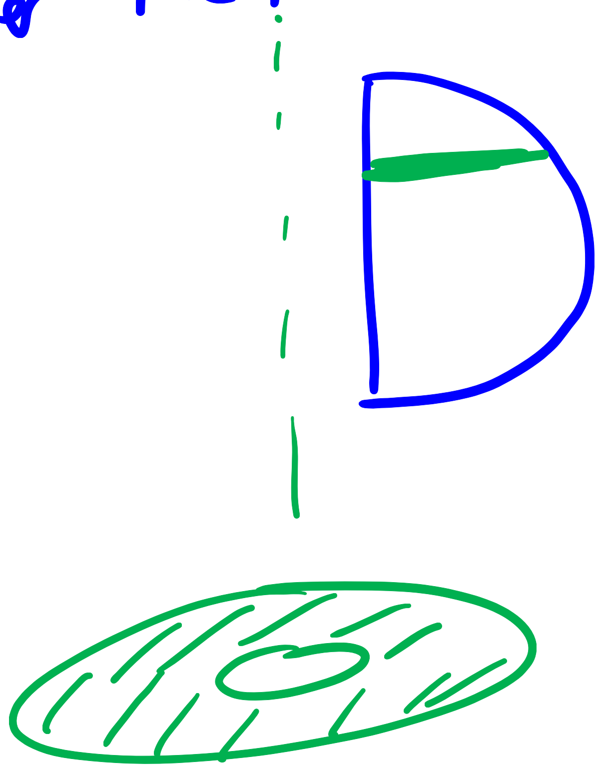
- Same way as before
  - → Integrating the cross sectional area over the length of the object.

Need to figure out the area

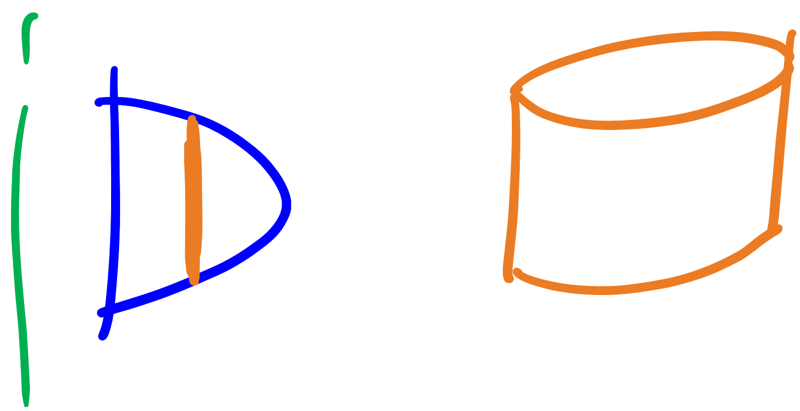
→ Can be tricky

Two Methods

Washer Method



Shell method

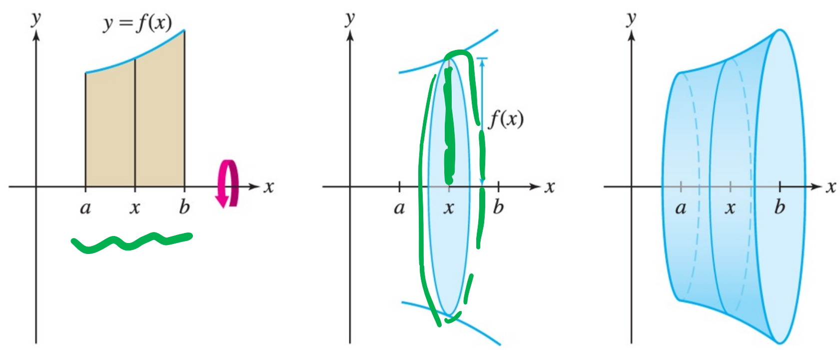


2 Disk Method

The first method for computing volumes of solids of revolution is the Disk Method, named this way because each cross-section is a disk, or circle.

How can we compute these volumes?

The idea is the same: Compute the area of each slice and 'add them up' (integrate) to get the volume. What's the area of each slice?



(A) (B) Cross section is a circle of radius $f(x)$. (C) Solid of revolution

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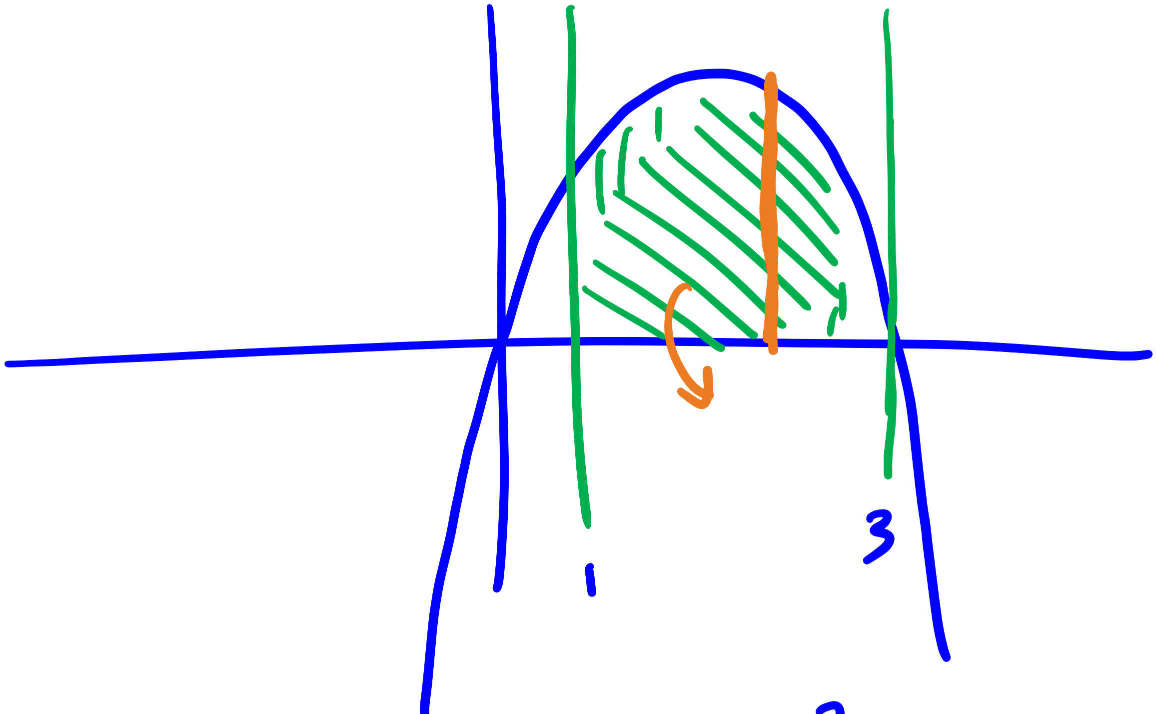
→ Every slice is a circle

$$A = \pi r^2 \Rightarrow A(x) = \pi f(x)^2$$

$$\text{Volume} = \int_a^b A(x)\,dx = \boxed{\int_a^b \pi f(x)^2\,dx}$$

Example: Find the volume of the solid obtained by rotating the region under the parabola $y = 6x - 2x^2$ between $x = 1$ and $x = 3$ around the x axis.



$$\text{Vol} = \int_1^3 \pi\left(6x - 2x^2\right)^2 dx$$

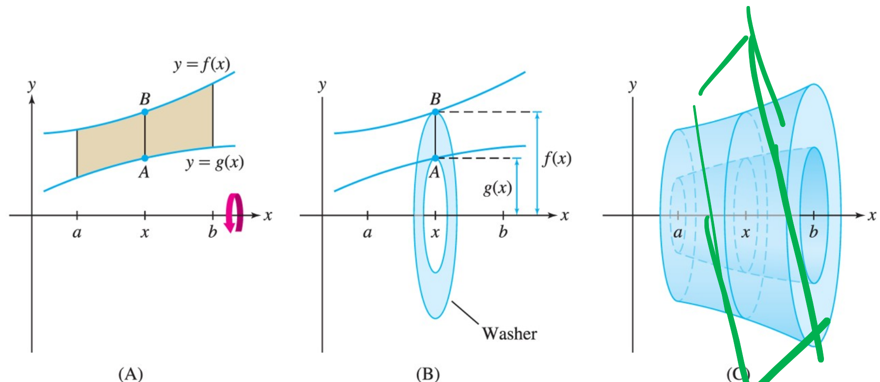
$$= \pi \int_1^3 36x^2 - 24x^3 + 4x^4 \, dx$$

$$= \pi\left(12x^3 - 6x^4 + \frac{4}{5}x^5\right)\Bigg|_1^3$$

$$\boxed{= \frac{128}{5}\pi}$$

3 Washer Method

There are shapes that are slightly more complicated that can also be analyzed fairly easily by this method. The idea is similar to going from 'Area under a curve' to 'Area between two curves.'



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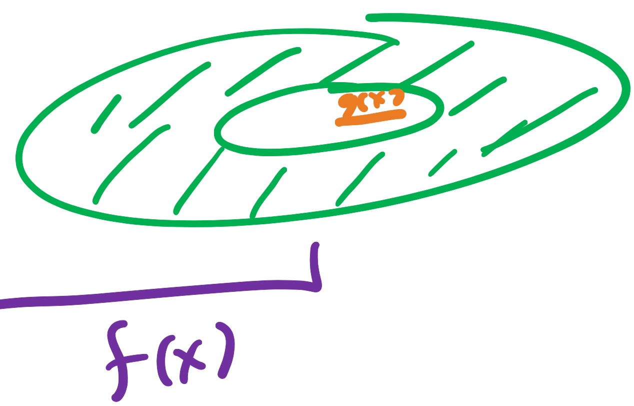
What do the slices look like here?



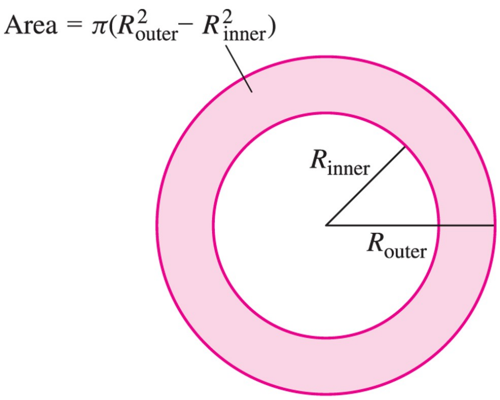
→ Need the area of each of these slices

→ Integrate to find volume of solid.

What is the area of each slice here?

Washer - Outer radius f

- Inner radius g.



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$$\text{Area} = \pi R_{\text{outer}}^2 - \pi R_{\text{inner}}^2$$

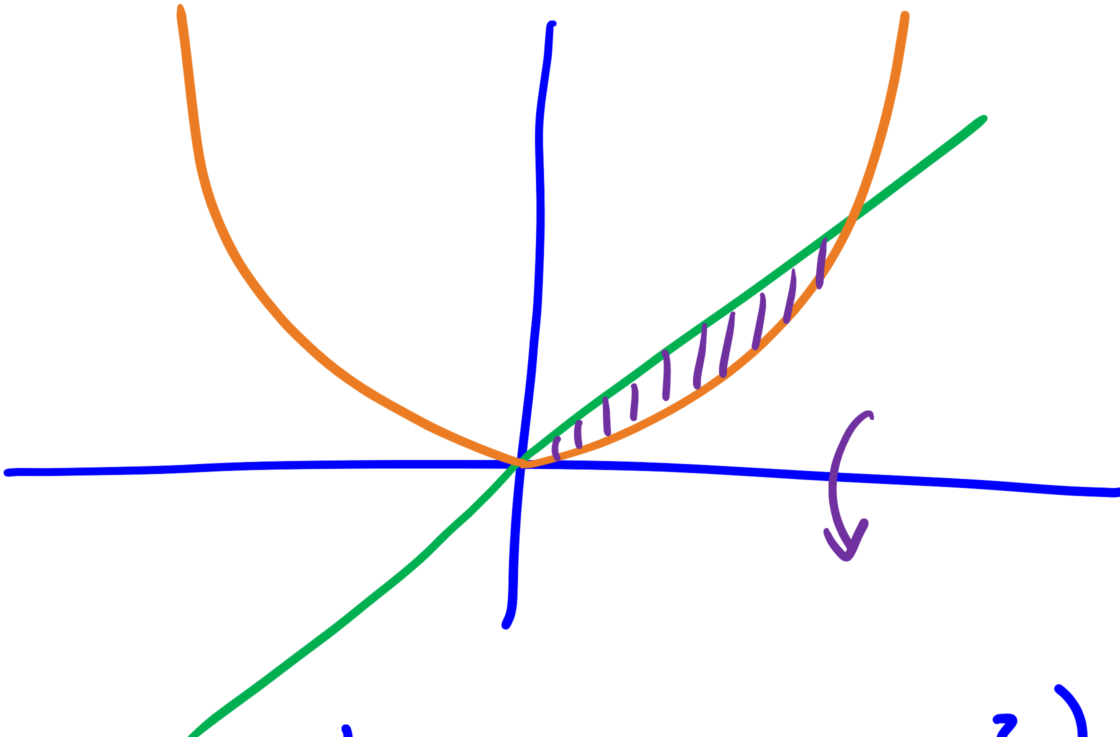
$$\text{Area}(x) = \pi f(x)^2 - \pi g(x)^2$$

$$= \pi\left(f(x)^2 - g(x)^2\right)$$

$$\neq \cancel{\pi\left(f(x) - g(x)\right)^2}$$

$$\boxed{\text{Volume} = \int_a^b \pi\left(f(x)^2 - g(x)^2\right)\,dx}$$

Example: Find the volume of the solid of revolution obtained by revolving the region between the curves $y = x$ and $y = x^2$ around the x-axis.



$x=0,\ x=1$

$$V = \int_0^1 \pi\left(\text{outer}^2 - \text{inner}^2\right) dx$$

$$= \pi \int_0^1 x^2 - (x^2)^2 \, dx$$

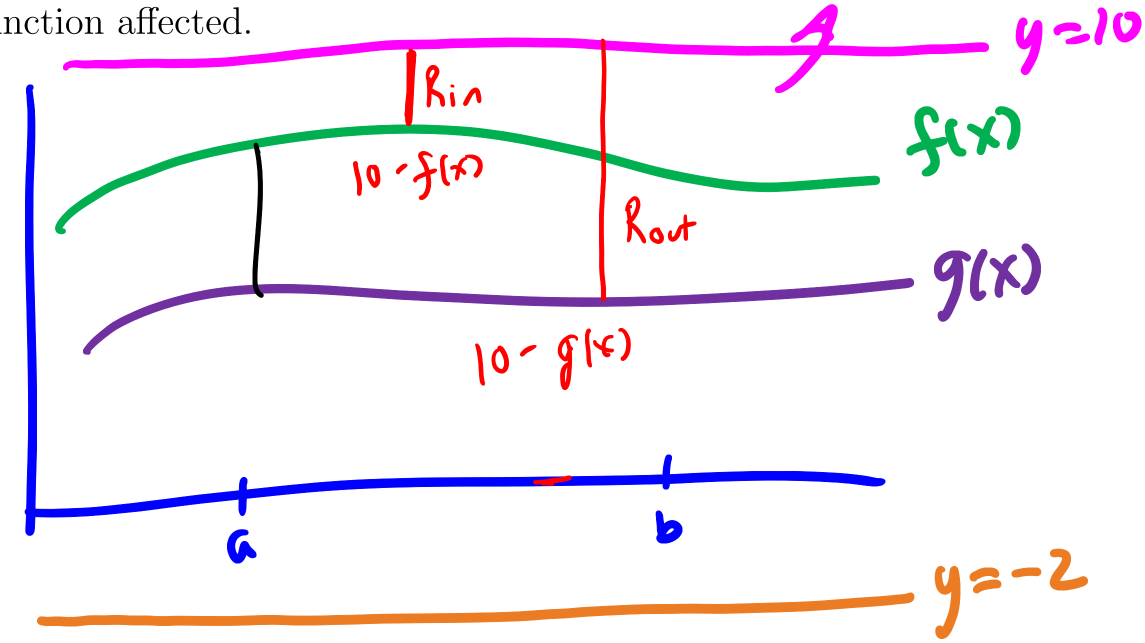
$$= \pi \int_0^1 x^2 - x^4 \, dx$$

$$= \pi\left(\frac{x^3}{3} - \frac{x^5}{5}\right)\Big|_0^1 = \pi\left(\frac{1}{3} - \frac{1}{5}\right)$$

$$\boxed{= \frac{2\pi}{15}}$$

4 Rotation around other lines

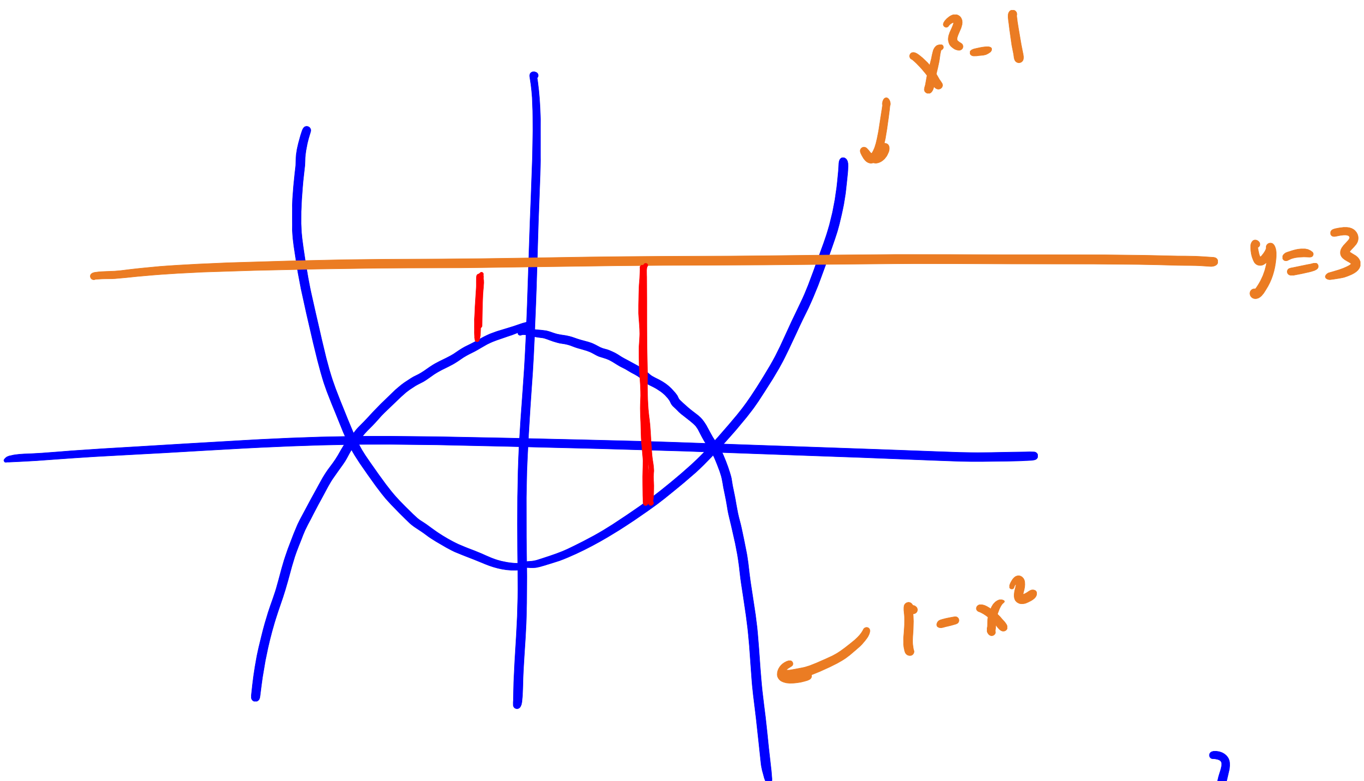
It is also to find the volume of solids of revolution when regions are revolved around different horizontal lines. The main thing to figure out is how is the 'radius' of each function affected.



$$\text{Vol} = \pi \int_a^b (f(x)+2)^2 - (g(x)+2)^2 \, dx \qquad y=-2$$

$$\text{Vol} = \pi \int_a^b (10-g(x))^2 - (10-f(x))^2 \, dx \qquad y=10$$

Example: Find the volume of the solid of revolution obtained by revolving the region between $y = 1 - x^2$ and $y = x^2 - 1$ around the line $y = 3$.



$$V = \pi \int_{-1}^{1} \left(3 - (x^2-1)\right)^2 - \left(3 - (1-x^2)\right)^2 \, dx$$

$$= \pi \int_{-1}^{1} (4-x^2)^2 - (2+x^2)^2 \, dx$$

$$= \pi \int_{-1}^{1} 16 - 8x^2 + \cancel{x^4} - (4 + 4x^2 + \cancel{x^4}) \, dx$$

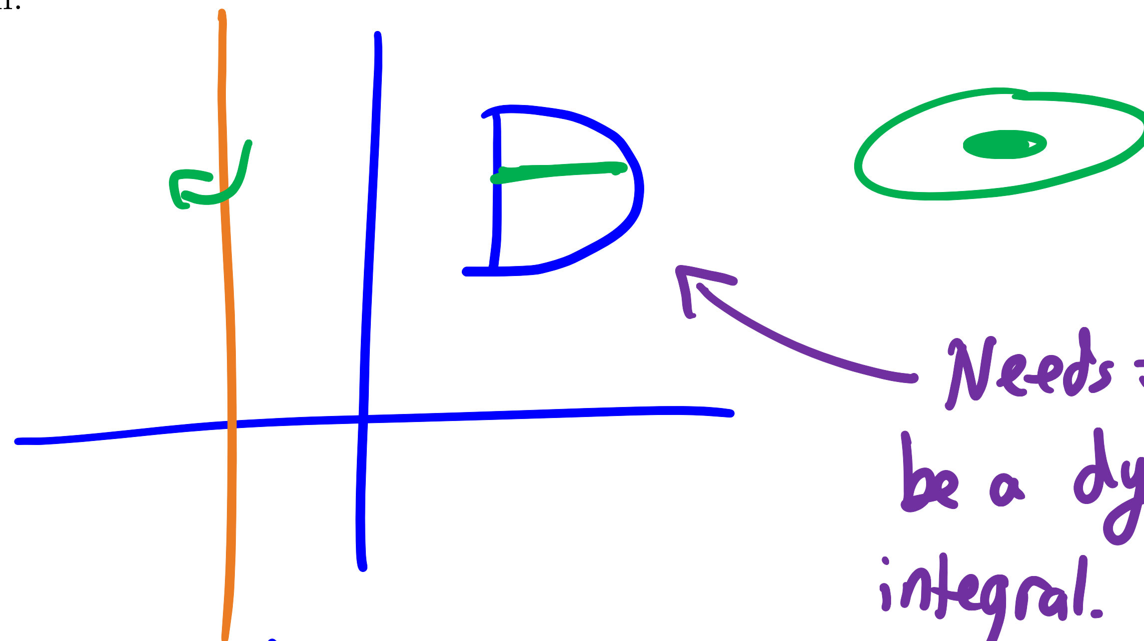
$$= \pi \int_{-1}^{1} 12 - 12x^2 \, dx = \pi \left(12x - 4x^3\right)\Big|_{-1}^{1}$$

$$= \pi\left((12-4) - (-12+4)\right)$$

$$= \boxed{16\pi}$$

5 Rotation around vertical lines

All of these types of problems can also be done with rotation around vertical lines as well.

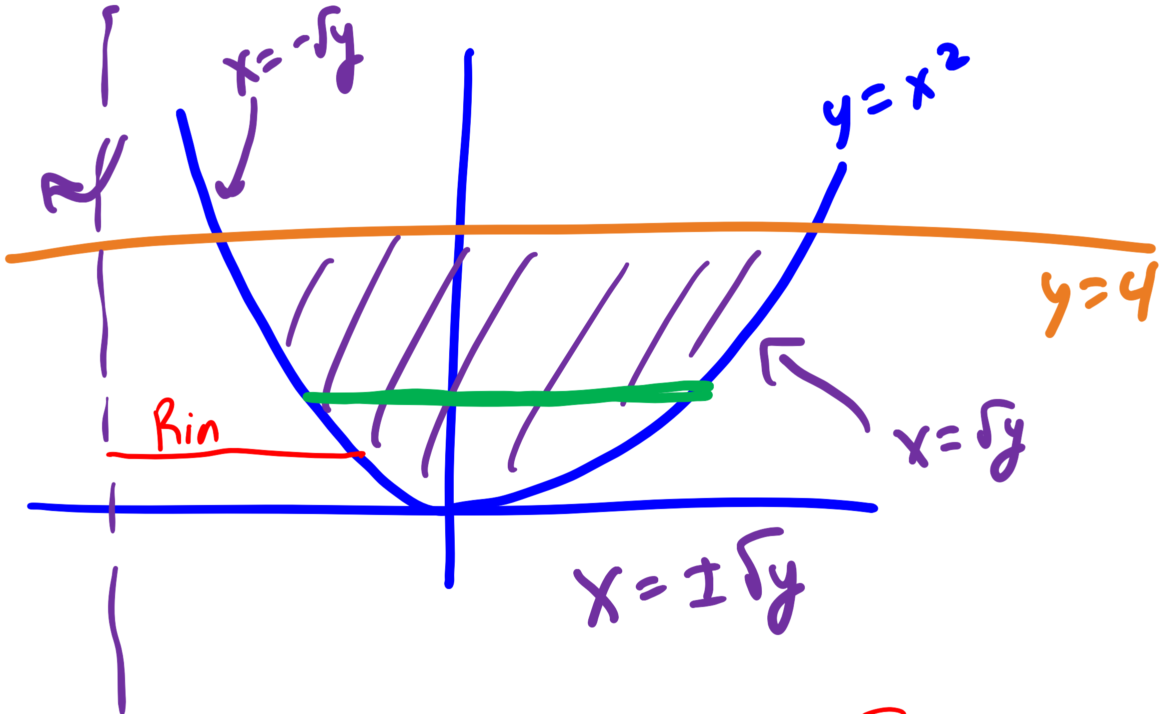


Needs to be a dy integral.

$$Vol = \pi \int_c^d (f(y))^2 - (g(y))^2 \, dy$$

→ Need to write x as a function of y to make this work.

Example: Find the volume of the solid of revolution obtained by revolving the region between the curves $y = x^2$ and $y = 4$ around the line $x = -3$.



$$R_{in} = -\sqrt{y} - (-3) = 3 - \sqrt{y}$$

$$R_{out} = \sqrt{y} - (-3) = 3 + \sqrt{y}$$

$$\text{Vol} = \pi \int_0^4 (3+\sqrt{y})^2 - (3-\sqrt{y})^2 \, dy$$

$$= \pi \int_0^4 (\cancel{9} + 6\sqrt{y} + \cancel{y}) - (\cancel{9} - 6\sqrt{y} + \cancel{y}) \, dy$$

$$= \pi \int_0^4 12\sqrt{y} \, dy = \overset{4}{\cancel{12}}\pi \cdot \frac{2}{\cancel{3}} y^{3/2} \Big|_0^4 = \boxed{64\pi}$$