Setting Up Integrals

Learning Goals

- Determine when certain quantities can be expressed as integrals
- Find the volume of an object given the cross-sectional area
- Find the volume of simple geometric objects using integrals
- Compute mass of objects given linear density
- Find the average value of an integrable function on an interval

Contents

1 Applications of Integrals

Some known applications of integrals:

- Net change from rate of change
- Area between curves

What else can be done with integrals?

→ Always result in the "total amount of stuff"

→ Anything that involves a "total amount" or adding things up can be done with an integral

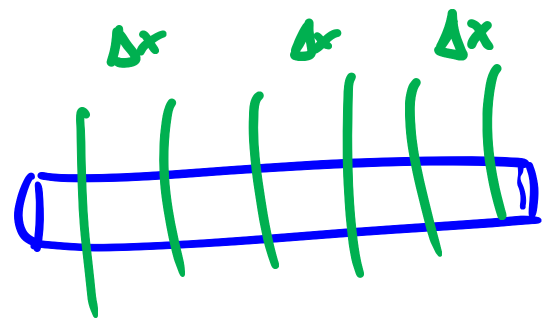
Examples of this:

- Total mass of the object by "adding up" little pieces.
- Total Volume of a solid by "adding up" small volumes.

Warning about Units:

For these problems, units can also be analyzed to determine how to solve it. It's a helpful tool, but care needs to be taken in how it is applied.

Total Mass



$$M = \sum_{i=1}^{10} m_i$$

$$= \sum_{i=1}^{10} \rho_i \; \underline{\Delta x}$$

cm, in

$\frac{g}{cm}$, $\frac{lb}{in}$

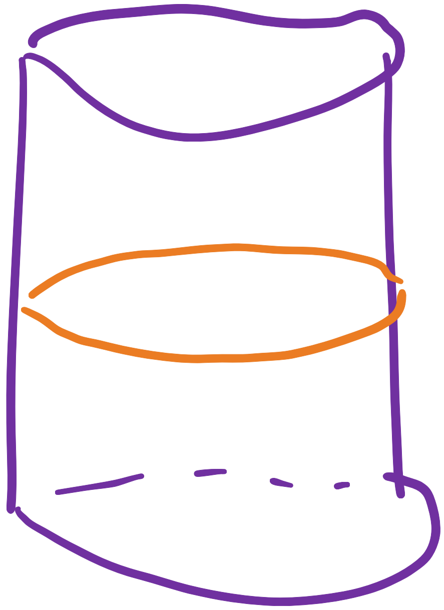
$$V = \sum_{i=1}^{10} V_i$$

$$= \sum_{i=1}^{10} A_i \; \Delta x$$

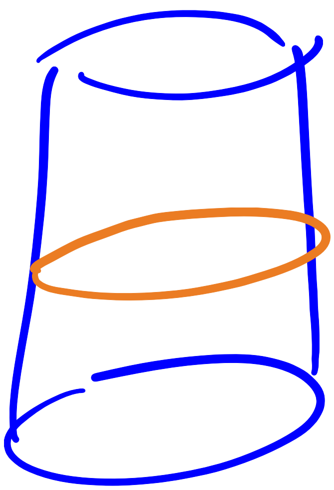
Area

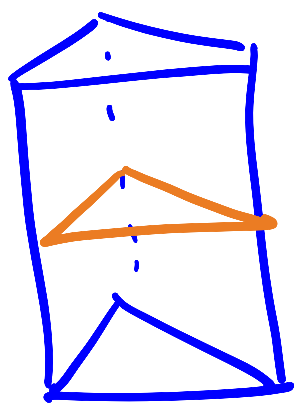
2 Volume of Simple Objects

The first application discussed here will be computing the volume of a solid from the cross-sectional area.



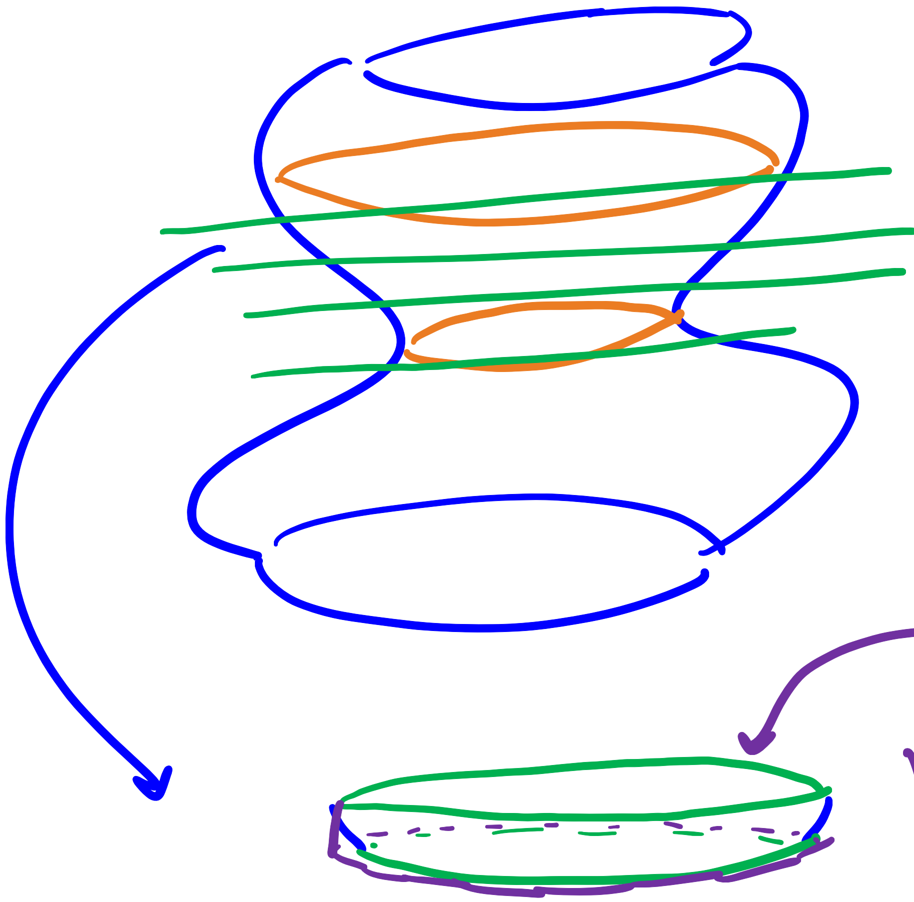
$V = A_b \cdot h$





$A_b \cdot h$

What happens if the area is not constant?



$V \approx A \cdot \Delta h$

→ Add all of these up to get total volume

$$V \approx \sum_{k=1}^{N} A_k \Delta h \rightarrow \boxed{V = \int_0^H A(h)\, dh}$$

→ Volume given cross-section area.

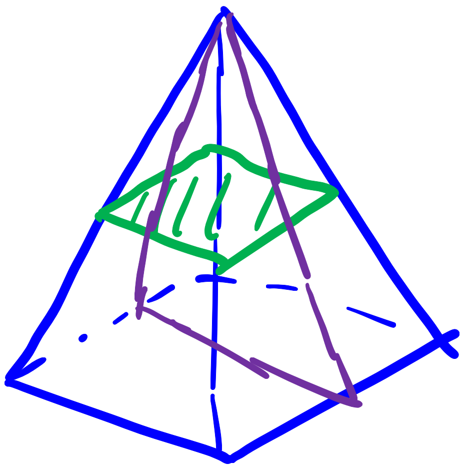
Units

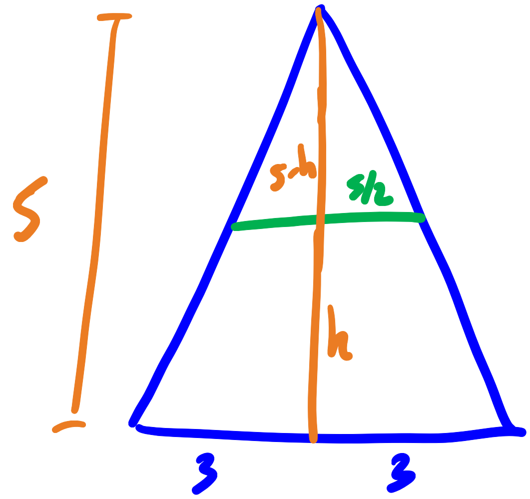
Why does this make sense from a unit context?

$$\underbrace{V}_{m^3} \approx \sum \underbrace{A_k}_{m^2} \underbrace{\Delta h}_{m}$$

$$\underbrace{V}_{m^3} = \int \underbrace{A(h)}_{m^2} \underbrace{dh}_{m}$$

Example: Find the volume of a pyramid with square base of height 5 m and base side length 6 m.





Similar triangles $\frac{3}{5} = \frac{s/2}{5-h}$

$$15 - 3h = 5\, s/2$$

$$\frac{30-6h}{5} = s = \underbrace{6 - \frac{6}{5}h}$$

$$V = \int_0^5 \overset{s^2}{A(h)}\, dh = \int_0^5 (6 - 6/5\, h)^2\, dh$$

$$= \int_0^5 36 - \frac{72}{5}h + \frac{36}{25}h^2\, dh = 36h - \frac{36}{5}h^2 + \frac{12}{25}h^3$$

$$= \boxed{60}$$

3 Volume from Cross-Sectional Area

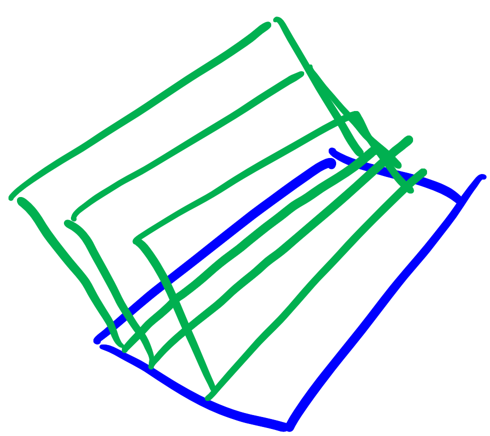
How else can solids be described for this method?

Need:

- Where my object is?
- What each cross-section looks like in order to find its area.

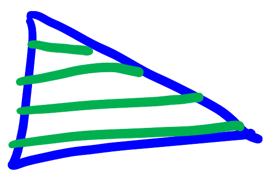
Could be given:

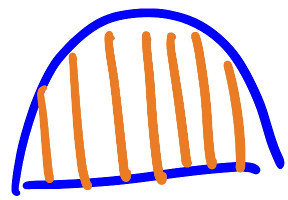
- A base for where the object sits
- What the cross sections look like.



Things to keep track of:

- Which way do the cross-sections go?
  - → This will tell you if you want dx or dy integrals

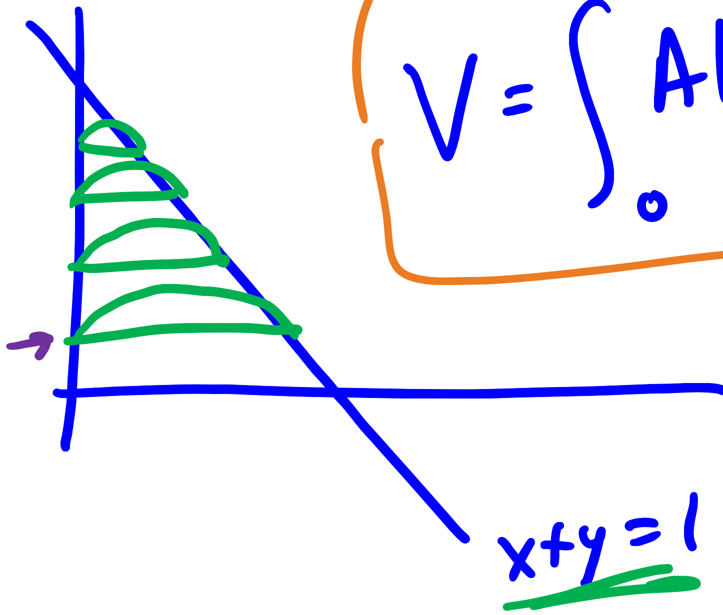




  - → Make sure you convert everything to the correct variable to solve

- What are the bounds of integration?

Example: Compute the volume of the solid whose base is the triangle enclosed by $x + y = 1$, the x-axis, and the y-axis. The cross sections perpendicular to the y-axis are semicircles.

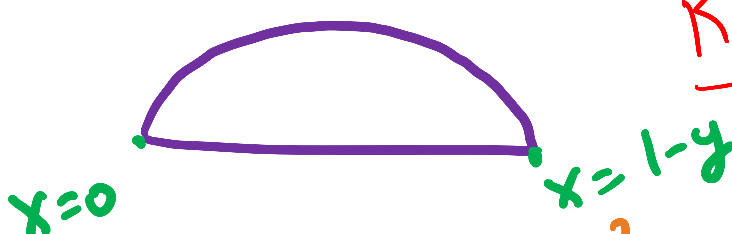


$$\boxed{V = \int_0^1 A(y)\, dy}$$

What is $A(y)$?

Semicircle → Need radius

Radius $= \frac{1-y}{2}$



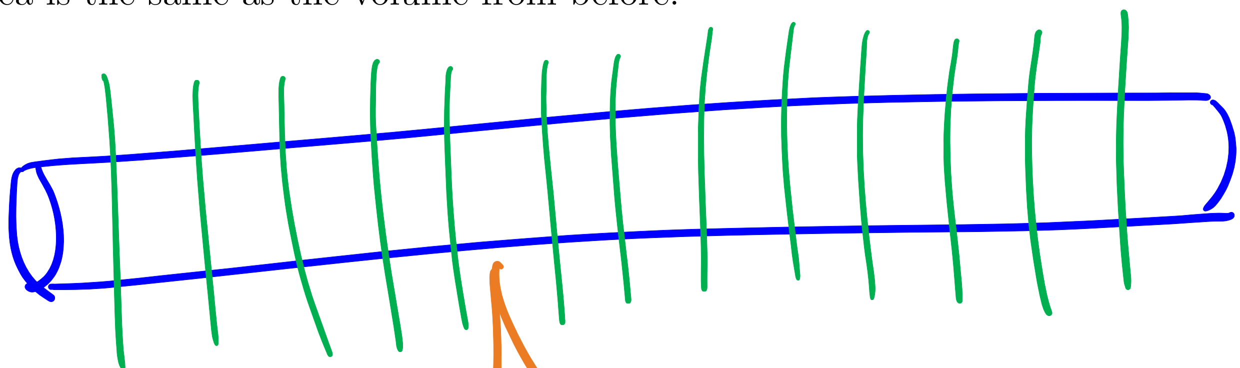
$$A(y) = \frac{\pi}{2}\left(\frac{1-y}{2}\right)^2 = \frac{\pi}{8}(1-y)^2$$

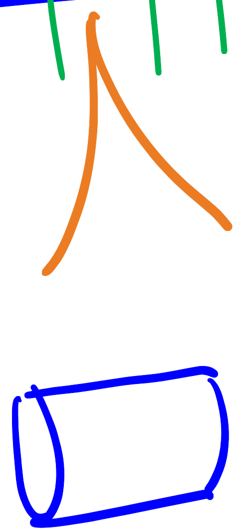
$$V = \int_0^1 \frac{\pi}{8}(1-2y+y^2)\, dy = \frac{\pi}{8}\left(y - y^2 + \frac{y^3}{3}\right)\Big|_0^1$$

$$= \boxed{\pi/24}$$

4 Mass of Objects

Integrals can also be used to find masses of objects with varying density. The idea is the same as the volume from before.





$$\text{Mass} = \rho \cdot \Delta x$$

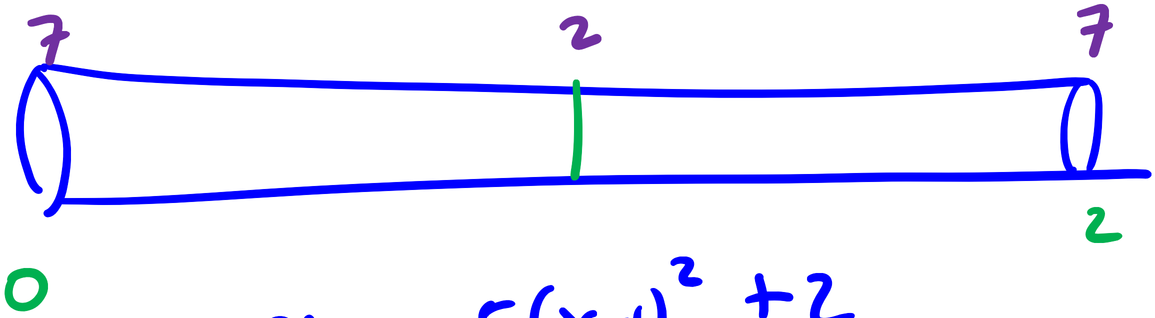
ρ – linear density

mass/length

Total Mass: $\displaystyle\sum_{k=1}^{N} \rho_k \cdot \Delta x$

$$\boxed{\text{Actual Mass} = \int_a^b \rho(x)\, dx}$$

Example: Assume that a 2m long metal rod has density given by $5(x-1)^2+2$ where x is distance in meters from one end of the rod. What is the total mass of this rod?



$$\rho(x) = 5(x-1)^2 + 2$$

$$\int_0^2 5(x-1)^2 + 2 \, dx$$

$$= \int_0^2 5x^2 - 10x + 5 + 2 \, dx$$

$$= \frac{5}{3}x^3 - 5x^2 + 7x \Big|_0^2$$

$$= \frac{5}{3}(8) - 5 \cdot 4 + 7 \cdot 2 = \boxed{\frac{22}{3}}$$

5 Average Value of Functions

One last application of integrals is to average value of functions. What is the average value of a set of numbers?

Numbers $\{a_1, a_2, \ldots, a_N\}$

$$\text{Average} = \frac{a_1 + a_2 + \cdots + a_N}{N}$$

Formula for Average value of a function

If f is an integrable function on $[a,b]$ its average value is

$$\boxed{\frac{1}{b-a}\int_a^b f(x)\,dx.}$$

Why is this an average value?

$$\underbrace{(\text{Ave Val } f)(b-a)}_{\text{Area of a rectangle}} = \underbrace{\int_a^b f(x)\,dx}_{\text{Area under the graph of } f.}$$

→ Average value is the height of the rectangle containing the same area of f.

→ "Average height" of the function.

Mean Value Theorem for Integrals

Theorem

If f is a continuous function on $[a, b)$, there is a c between a and b so that

$$F'(c) = f(c) = \frac{1}{b-a} \int_a^b f(x)\, dx$$

$F(b) - F(a)$

→ IVT – Average height must be hit.

→ MVT – Apply to antiderivative of f

Example: Find the average value of the function $f(x) = x^2 - 2$ on $[0, 3]$.

$$\text{Ave Val} = \frac{1}{3-0}\int_0^3 (x^2-2)\,dx$$

$$= \frac{1}{3}\left(\frac{x^3}{3} - 2x\right)\Big|_0^3$$

$$= \frac{1}{3}\left(\left(\frac{27}{3} - 6\right) - 0\right) = \boxed{1}$$