Area between Two Curves

Learning Goals

- Compute the area of vertically simple regions in the plane
- Compute the area of horizontally simple regions in the plane
- Recognize regions as vertically simple or horizontally simple, or split regions into ones that have this property
- Compute the area bounded by two curves
- Set up and compute areas of regions where multiple integrals are required

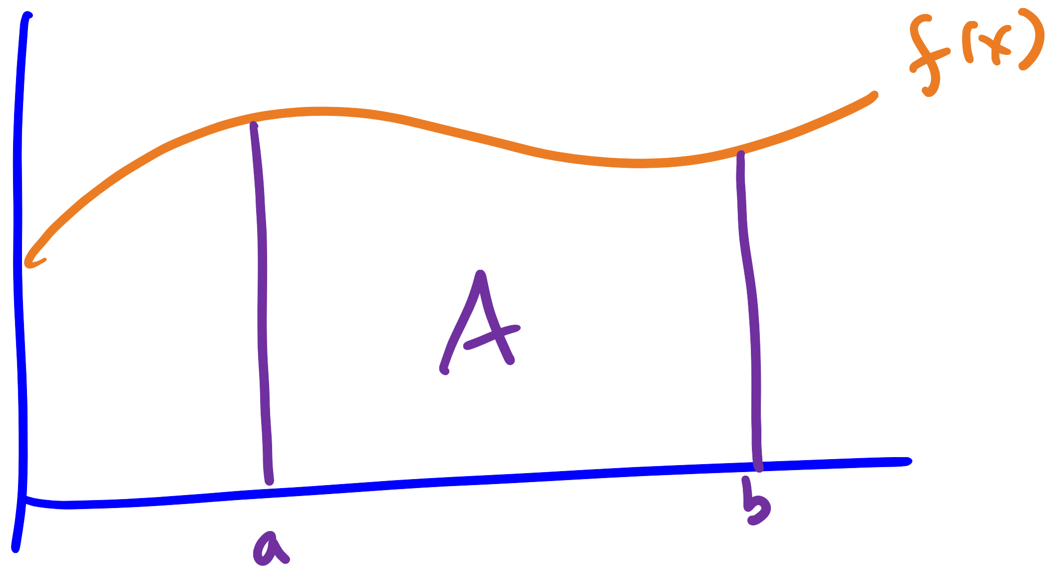
Contents

1 Introduction

Area between two curves is a first application of integration, of which there will be many more. The general idea is finding areas of regions in the plane, some of which are easier than others.

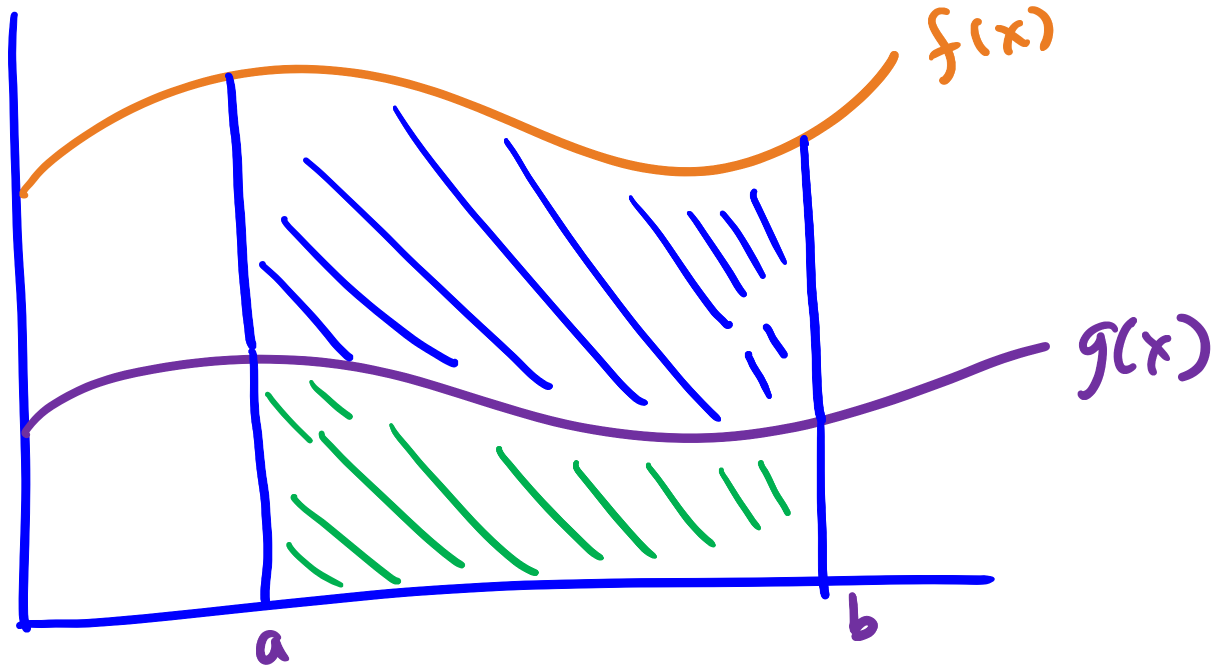
Easiest Type of Region

The definition of an integral gives us an easy way to find the area of very specific regions in the plane.



$$A = \int_a^b f(x)\,dx$$

We can do something a little more general as well.



Area: $\int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

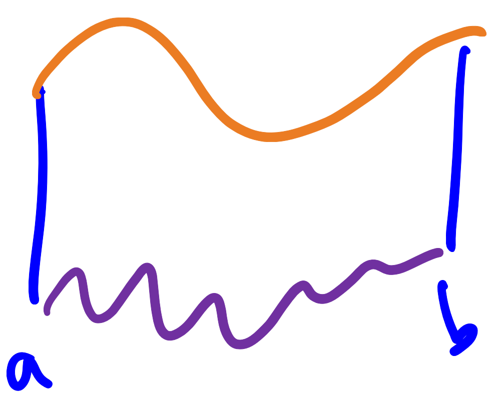
$$\boxed{A = \int_a^b f(x) - g(x)\,dx}$$

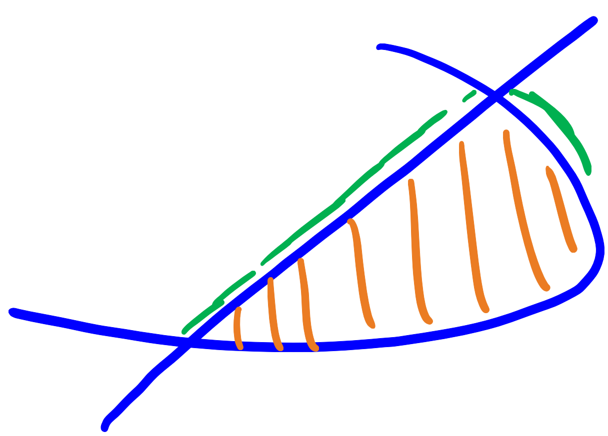
Regions like this are called **vertically simple**.

→ Single function on top, single function below for the entire region

That is, there are functions $f(x)$ and $g(x)$ so that the region is

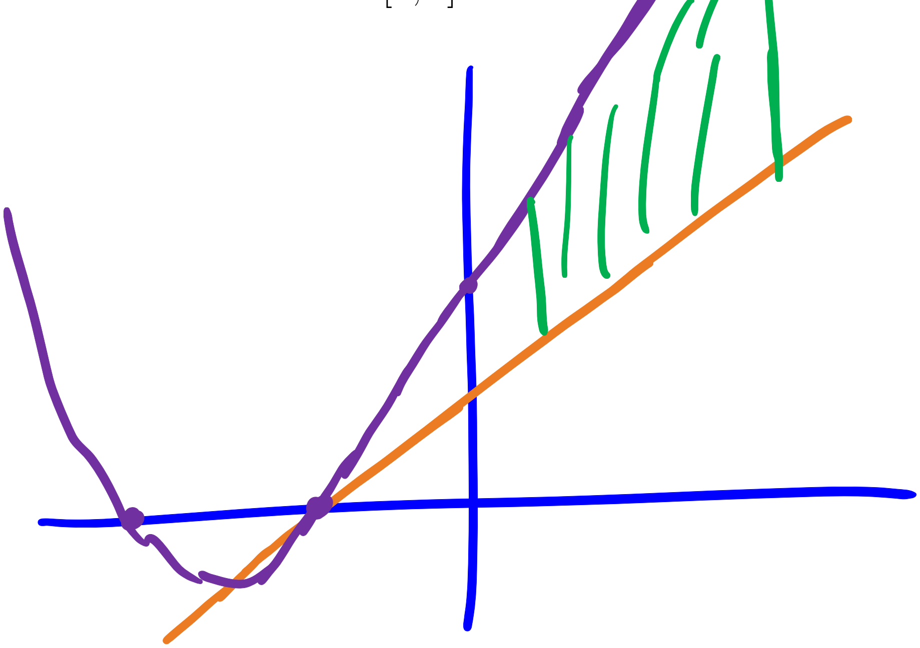
$$(x,y) \quad a \le x \le b, \quad g(x) \le y \le f(x)$$





Example: Find the area between the graphs of $y = x + 1$ and $y = x^2 + 3x + 2$ over the interval $[1, 4]$.

$(x+2)(x+1)$



$y = 2 + 1 = 3$

$y = 4 + 6 + 2 = 12$

$$\int_1^4 (x^2 + 3x + 2) - (x + 1) \, dx$$

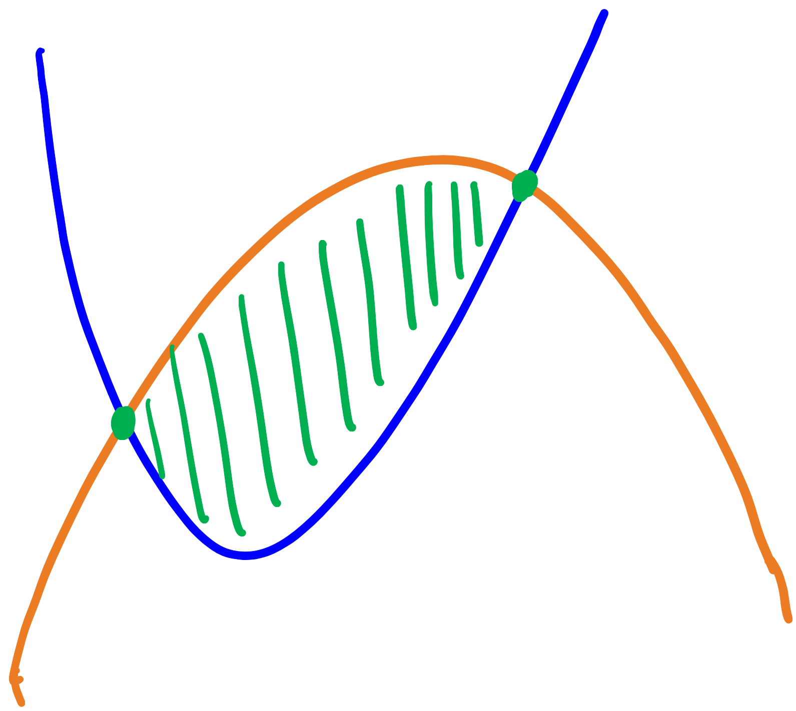
$$\int_1^4 x^2 - 2x + 1 \, dx = \frac{x^3}{3} - x^2 + x \Big|_1^4$$

$$= \left(\frac{64}{3} - 16 + 4\right) - \left(\frac{1}{3} \cancel{-1+1}\right)$$

$$21 - 16 + 4 = \boxed{9}$$

2 Region Bounded by Two Curves

Another type of question that can be answered by these methods is finding the area of the region bounded between two curves.

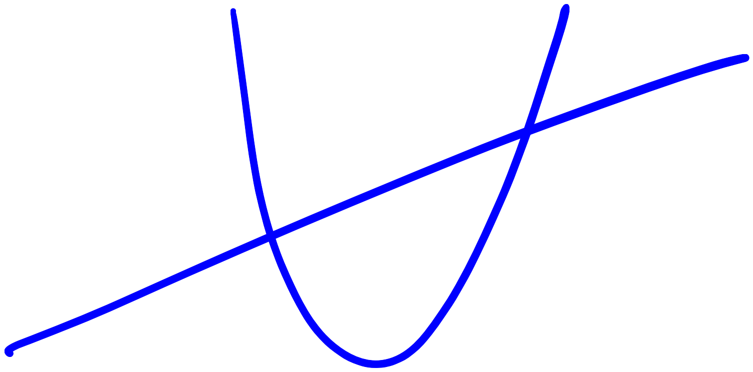


- Find intersection points to determine where the curves cross.
- Figure out which function is on top in this region.
- Work out the integral like normal.

Example: Find the area of the region bounded by the graphs of $y = x^2 + 2$ and $y = 3x + 6$.

top

bottom



Intersection points

$x=4,\ x=-1$

$$x^2+2 = 3x+6$$

$$x^2-3x-4 = 0$$

$$(x-4)(x+1) = 0$$

$$\int_{-1}^{4} (3x+6) - (x^2+2)\ dx$$

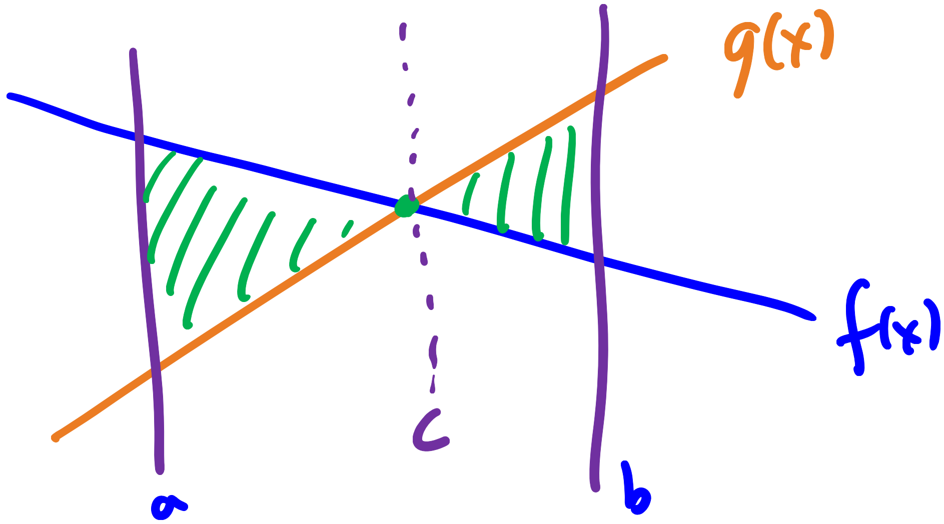
$$= \int_{-1}^{4} 3x+4-x^2\ dx = \frac{3x^2}{2} + 4x - \frac{x^3}{3}\Big|_{-1}^{4}$$

$$= \frac{48}{2} + 16 - \frac{64}{3} - \left(\frac{3}{2} - 4 + \frac{1}{3}\right)$$

$$= 24 + 16 - \frac{65}{3} + 4 - \frac{3}{2} = \boxed{\frac{125}{6}}$$

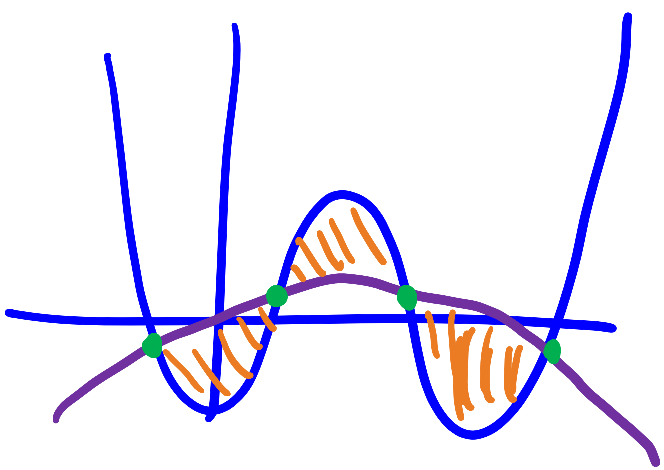
3 Curves that Cross Multiple Times

There can also be situations where the area of a different type of region is needed. This can arise from either an interval being given, and the two curves crossing in the middle, or the area bounded by two curves that cross multiple times.



$$\cancel{\int_a^b f(x)-g(x)\,dx}$$

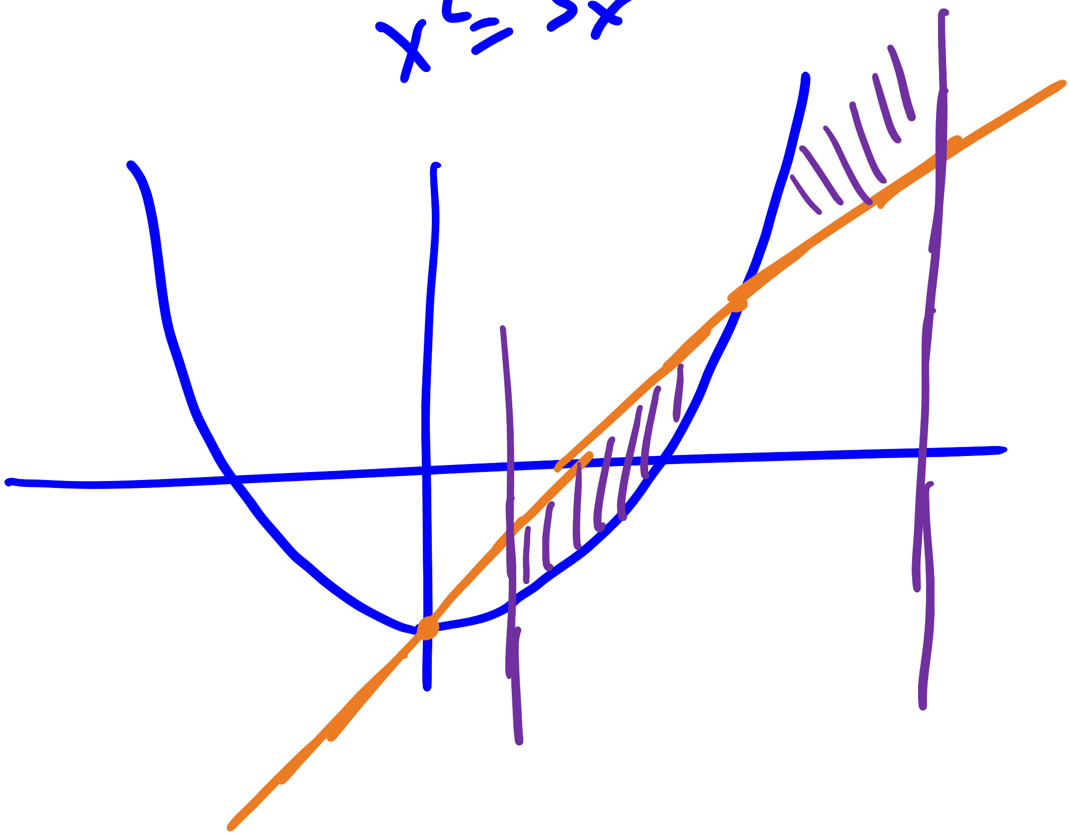
$$\int_a^c f(x)-g(x)\,dx + \int_c^b g(x)-f(x)\,dx$$



Example: Find the area between the graphs of $y = x^2 - 4$ and $y = 3x - 4$ over the interval $[1, 5]$.

$$x^2 - 4 = 3x - 4$$

$$x^2 = 3x \qquad x = 0 \text{ or } 3$$



$$\int_1^3 (3x-4) - (x^2-4)\, dx + \int_3^5 (x^2-4) - (3x-4)\, dx$$

$$\int_1^3 3x - x^2\, dx + \int_3^5 x^2 - 3x\, dx$$

$$= \frac{3x^2}{2} - \frac{x^3}{3}\Big|_1^3 + \frac{x^3}{3} - \frac{3x^2}{2}\Big|_3^5 =$$

$$\left(\frac{27}{2}-\frac{27}{3}\right)-\left(\frac{3}{2}-\frac{1}{3}\right)+\left(\frac{125}{3}-\frac{75}{2}\right)$$

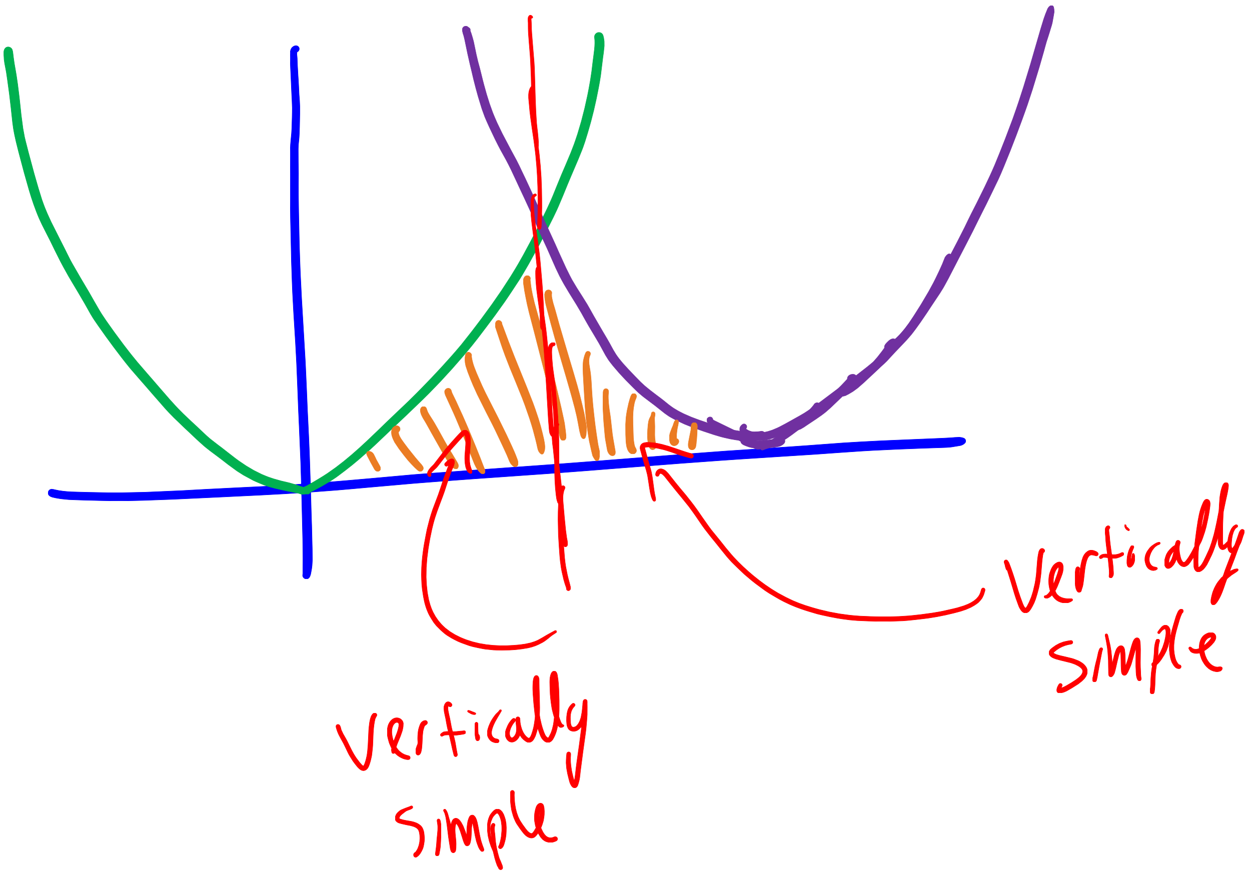
$$-\left(\frac{27}{3}-\frac{27}{2}\right)$$

$$\frac{27}{6}-\frac{7}{6}+\frac{25}{6}+\frac{27}{6}$$

$$=\frac{72}{6}=12$$

4 Multiple Vertically Simple Regions

Sometimes, it is not so easy to find the area of a region using an integral. The reason the regions discussed previously were so easy is because there was one function on the top and one function on the bottom over the entire region. That is not always the case.



→ Figure out which function is on top of the region when, and use that to determine the integrals you need.

Process for these Problems

For pretty much all cases, it is possible to split the region into different pieces that are vertically simple. This will require several different integrals to find the total area.

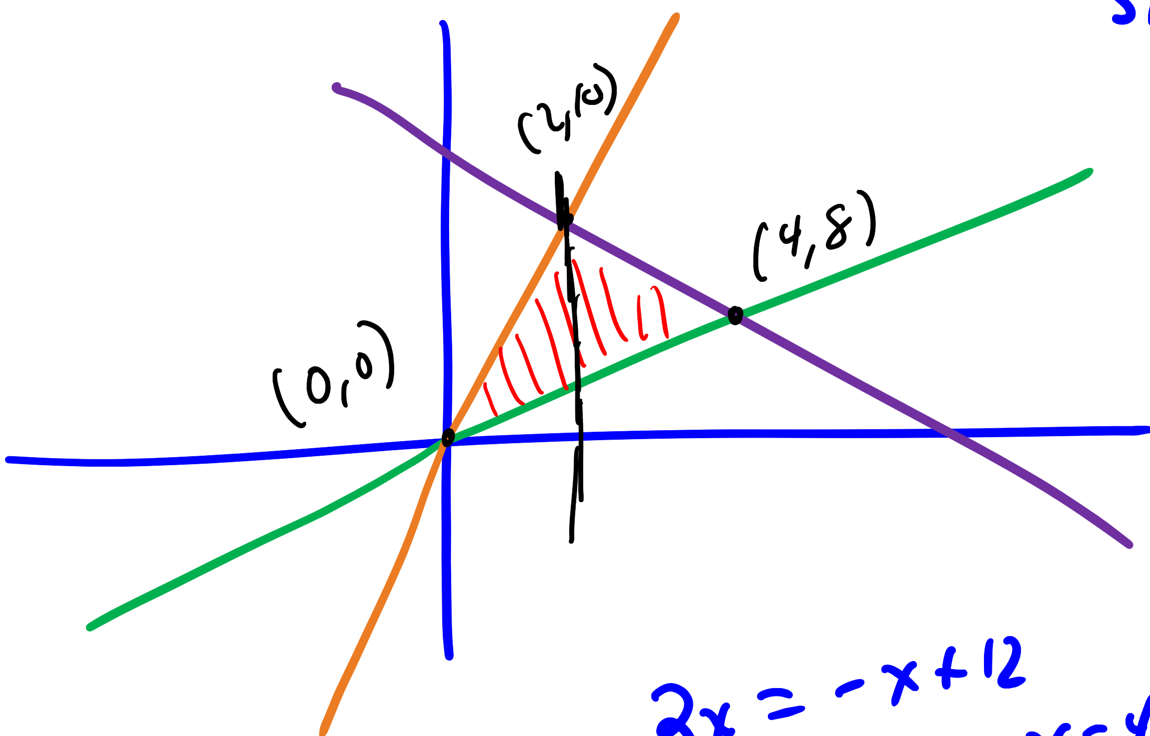
→ Figure out the curves that bound your region and figure out where they cross.

→ Use these intersection points to break your region into chunks, all of which are vertically simple.

→ Find the area of each of these chunks and use that to get your final answer.

Example: Find the area bounded between the graphs of $y = 2x$, $y = 5x$, and $y = -x + 12$

$5x = -x + 12$
$6x = 12$
$x = 2$



$2x = -x + 12$
$3x = 12$
$x = 4$

Total Area $= \int_0^2 5x - 2x \, dx + \int_2^4 (-x+12) - 2x \, dx$

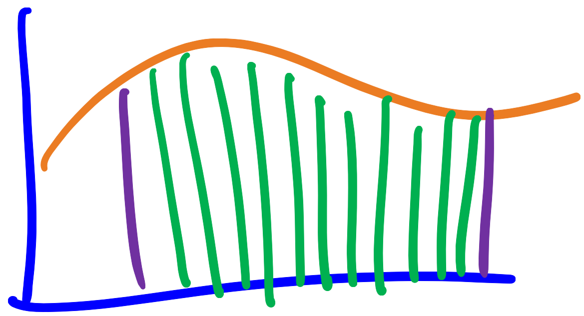
$\int_0^2 3x \, dx + \int_2^4 12 - 3x \, dx$

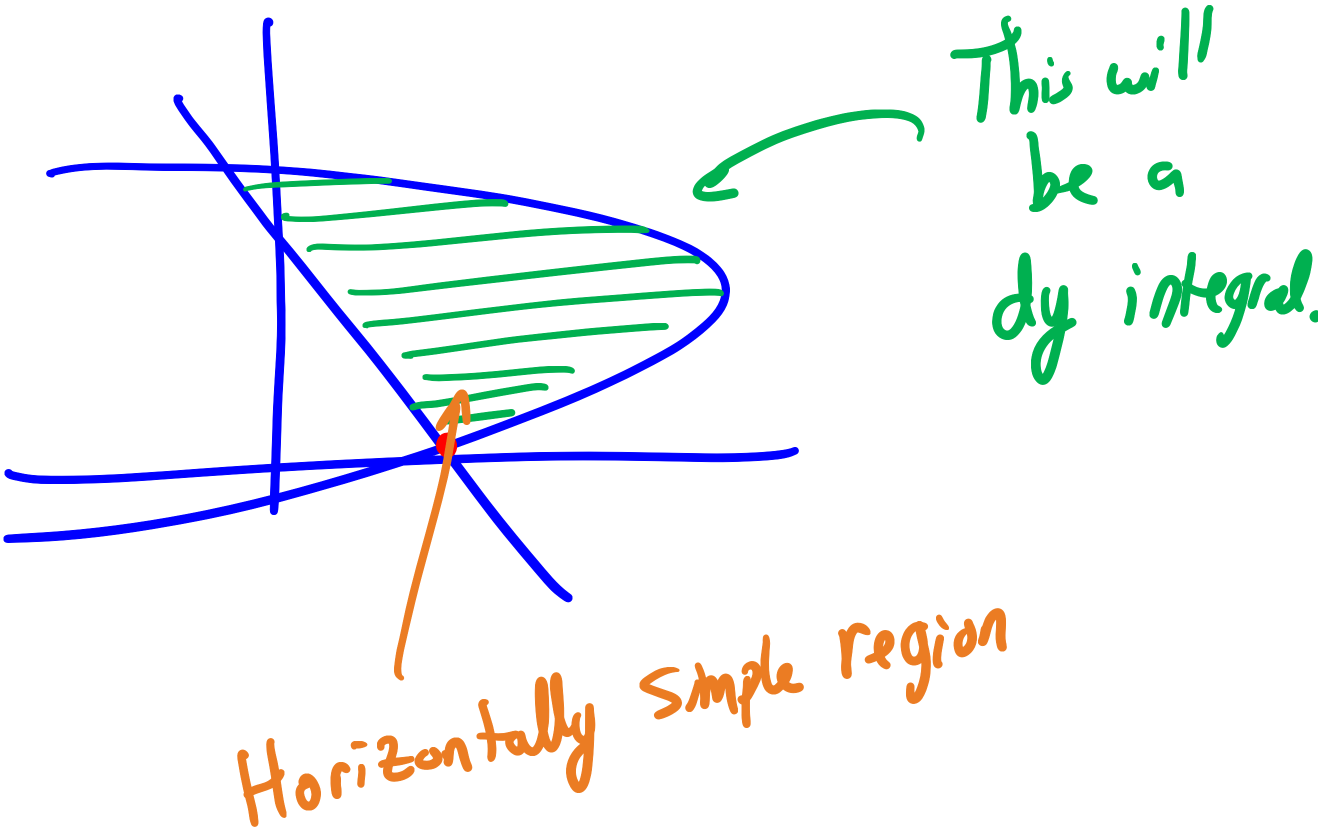
$\frac{3x^2}{2} \Big|_0^2 + 12x - \frac{3x^2}{2} \Big|_2^4 = 12$

$\frac{12}{2} + \left(48 - \frac{48}{2}\right) - \left(24 - \frac{12}{2}\right)$

5 Horizontally Simple Regions

The idea for finding area that has been developed so far has been 'adding up' (integrating) these small vertical lines to get the area under the curve. There's nothing saying this can't be done for horizontal lines as well.

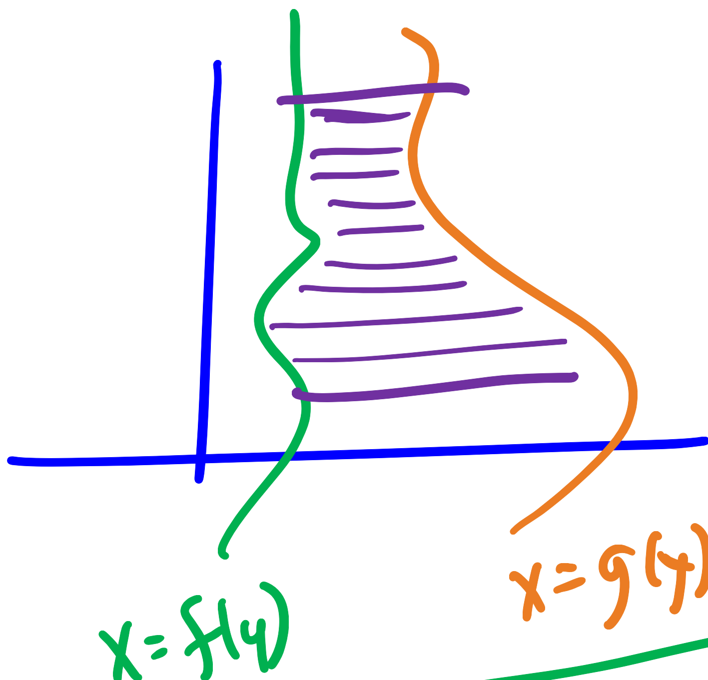




Definition: We say that a region is *horizontally simple* if there is a one curve on the left-side of the region and one curve on the right-side of the region over all of it.

Vertically simple: One on top one on bottom.

For a region like this, if we let $x = g(y)$ be the 'right' curve and $x = f(y)$ be the 'left' curve, then we can find the area of the region between $g(y)$ and $f(y)$ as



$$\boxed{A = \int_a^b x_{right} - x_{left} \, dy = \int_a^b g(y) - f(y) \, dy}$$

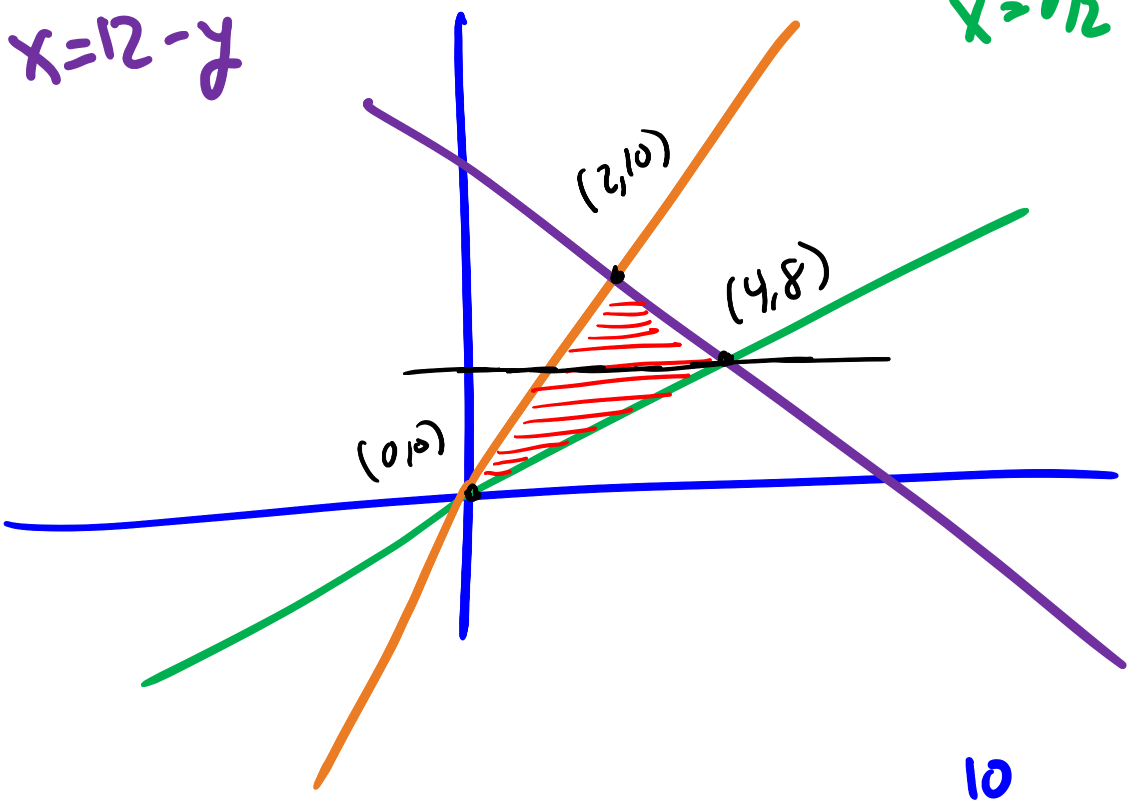
Ways this can be used

1. Regions where the boundaries are given as x equals a function of y
2. Regions that are horizontally simple and it is possible to solve out for x as a function of y.
3. Regions where it is easier to visualize the region and add it up using horizontal lines as opposed to vertical lines.

All of the methods discussed previously (multiple regions, curves crossing) can also be done for horizontally simple regions, or regions where you want to use y integrals.

Example: Find the area bounded between the graphs of $y = 2x$, $y = 5x$, and $y = -x + 12$ using integrals in y.

$x = 12 - y$ $\qquad$ $x = y/2$ $\qquad$ $x = y/5$



$$\text{Area} = \int_0^8 x_{\text{right}} - x_{\text{left}} \, dy + \int_8^{10} x_{\text{right}} - x_{\text{left}} \, dy$$

$$= \int_0^8 y/2 - y/5 \, dy + \int_8^{10} (12 - y) - \frac{y}{5} \, dy$$

$$= \int_0^8 \frac{3}{10} y \, dy + \int_8^{10} 12 - \frac{6y}{5} \, dy$$

$$= \frac{3y^2}{20}\Big|_0^8 + 12y - \frac{3y^2}{5}\Big|_8^{10}$$

$$= \frac{192}{20} - 0 + \left(120 - \frac{300}{5}\right) - \left(96 - \frac{192}{5}\right)$$

$$= \frac{48}{5} - 0 + 120 - 60 - 96 + \frac{192}{5}$$

$$= \frac{240}{5} + 60 - 96 = 12$$