The Substitution Method

Learning Goals

- Recognize when the substitution rule can be applied to integrals
- Compute indefinite integrals using substitution
- Compute definite integrals using substitution

Contents

1 Definition of the Method

Integration/antidifferentiation is much harder than differentiation. There aren't rules for computing integrals so much as there are 'techniques' for trying to solve the problem. One of these techniques is the *substitution method*.

The Chain Rule

The substitution method is essentially the inverse of the Chain Rule for differentiation.

$$f(x) = \cos(x^4)$$

$$f'(x) = -\sin(x^4) \cdot 4x^3$$

$$\int \underbrace{-4x^3 \sin(x^4)} \, dx = \cos(x^4) + C$$

Substitution Method

$$\int \underbrace{f(u(x)) \cdot u'(x)}\, dx = \underbrace{F(u(x))} + C$$

where $\underline{F \text{ is an antiderivative}}$ for f

i.e. $$\int f(x)\, dx = F(x) + C$$

$$\int \underbrace{4x^3}_{u'(x)} - \sin(\underbrace{x^4}_{u(x)})\, dx = \underline{\cos(x^4) + C}$$

Example: Compute

$$\int 2x \sin\left(x^2\right)\, dx$$

$u'(x) = 2x$

$u(x) = x^2$

$$\int \sin(u(x)) \cdot u'(x)\, dx = -\cos(u(x)) + C$$

$$= -\cos(x^2) + C$$

Substitution using Differentials

There is another way to think about substitution problems that is usually easier to implement. The idea is that we want to 'change variables' in the integral, so that it an integral in terms of u instead of x.

$$\underline{u(x)}$$

$$du = u'(x)\,dx$$

$$\int f(u(x))\,\underbrace{u'(x)\,dx} = \int f(u)\,du$$

$$= F(u) + C$$

$$= \underline{F(u(x)) + C}$$

Example: Compute

$$\int (x+2)(x^2+4x+10)^5 \, dx$$

$$u(x) = x^2+4x+10$$

$$du = 2x+4 \quad dx$$

$$dx = \frac{du}{2x+4}$$

$$\frac{1}{2} du = x+2 \ dx$$

$$\int (x+2)(u)^5 \cdot \frac{du}{2x+4}$$

$$\frac{1}{2}\int \cancel{(x+2)}\, u^5 \frac{du}{\cancel{x+2}} = \frac{1}{2}\int u^5 du$$

$$= \frac{1}{2}\frac{u^6}{6} + C$$

$$= \boxed{\frac{1}{12}(x^2+4x+10)^6 + C}$$

2 Definite Integrals by Substitution

What happens if we want to do the substitution method for a definite integral? The main thing is, we need to deal with the bounds, as well as where our function is defined.

$$\int_a^b f(u(x))\,u'(x)\,dx = \int_{u(a)}^{u(b)} f(u)\,du$$

$x=a$
$x=b$

We have $u(x)$

Formula for Definite Integrals

Assume that u' is continuous on $[a,b]$ and $f(u(x))$ is defined on $[a,b]$.

Then,

$$\boxed{\int_a^b f(u(x))\, u'(x)\, dx = \int_{u(a)}^{u(b)} f(u)\, du}$$

Example: Compute

$$\int_1^4 x(x^2+1)^3\,dx$$

$u(1) = 1^2+1 = 2$

$u(4) = 4^2+1 = 17$

$u(x) = x^2+1$

$du = 2x\,dx \rightarrow \frac{1}{2}\,du = x\,dx$

$$\frac{1}{2}\int_2^{17} u^3\,du = \frac{1}{2}\,\frac{u^4}{4}\Big|_2^{17}$$

$$= \boxed{\frac{1}{8}\left(17^4 - 2^4\right)}$$

3 Complicated Substitutions

Some of the hardest substitution problems to solve are ones that involve inverse trigonometric functions or exponentials. The issue here is it can be more difficult to spot what the 'inside' function should be and what type of expression you are aiming for when choosing u for these problems.

Choosing u:

→ 'Inside function'

→ Once you do this, you can integrate the result.

→ Can also look for du and use that to help.

$u = \cos\theta$

$du = -\sin\theta \, d\theta$

$$\int \tan\theta \, d\theta = \int \frac{\sin\theta \, d\theta}{\cos\theta}$$

$$= -\int \frac{1}{u} \, du = -\ln|\cos\theta| + C$$

Example: Compute

$$\int \frac{4x}{\sqrt{1-x^4}}\,dx$$

$1 - x^4 = u$

$du = -4x^3\,dx$

$u = x^2$

$du = 2x\,dx$

$$\int \frac{2\,du}{\sqrt{1-u^2}}$$

$= 2\sin^{-1}(u) + C$

$= \boxed{2\sin^{-1}(x^2) + C}$

Two more examples:

$$\int \frac{x}{16x^4+1}\,dx \qquad\qquad \int \frac{64x^3}{16x^4+1}\,dx$$

$u = 4x^2$

$du = 8x\ dx$

$$\frac{1}{8}\int \frac{du}{u^2+1} = \frac{1}{8}\tan^{-1}(4x^2)+C$$

$u = 16x^4+1$

$du = 64x^3\ dx$

$$\int \frac{1}{u}\,du$$

$$= \boxed{\ln|16x^4+1| + C}$$