Arc Length and Area in Polar Coordinates

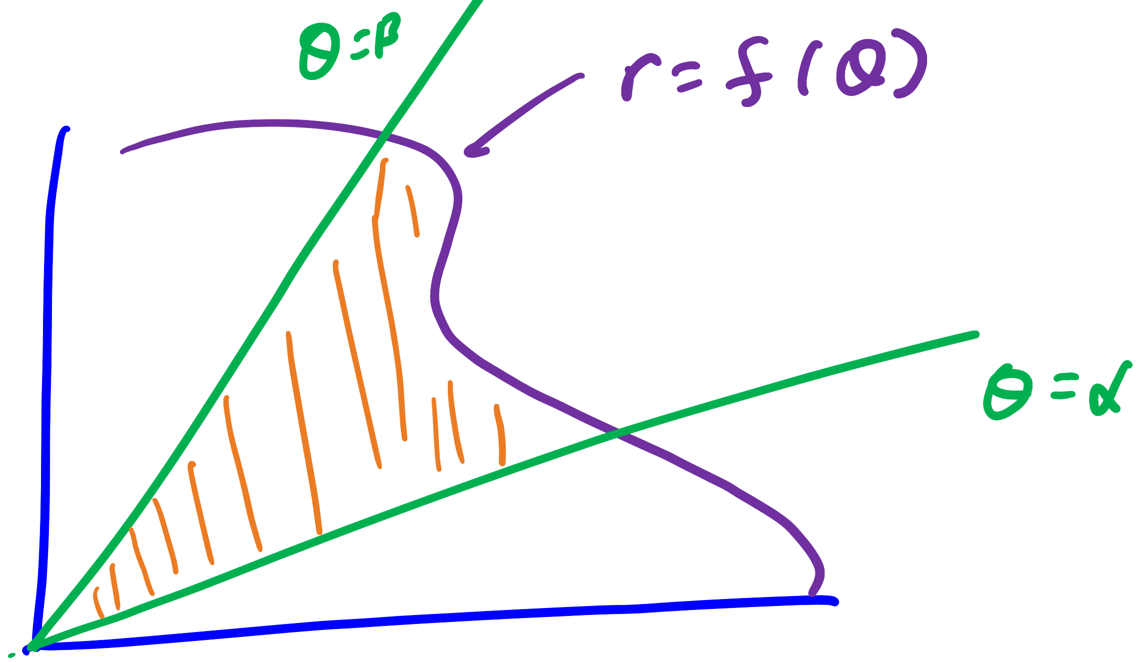
Learning Goals

- Find the area of a region bounded by a polar curve
- Find the area of a region between two polar curves
- Find the arc length of a polar curve

Contents

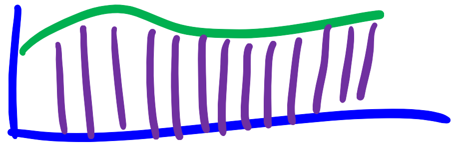
1 Area for Polar Functions

Assume that we write $r = f(\theta)$ with $f(\theta) > 0$ for $\alpha \leq \theta \leq \beta$. We want to find the area enclosed inside the graph and the sector between $\theta = \alpha$ and $\theta = \beta$.



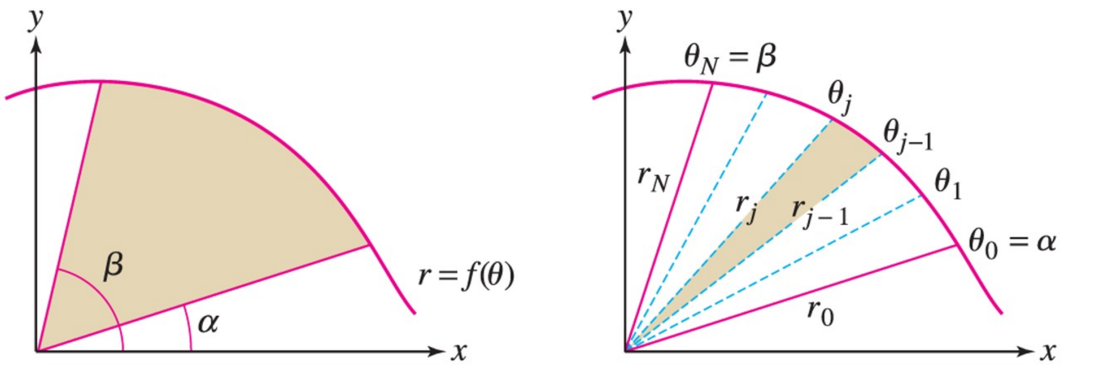
→ Would be very complicated in Cartesian coordinates.

→ Polar makes this easier.



How do we find this area here?

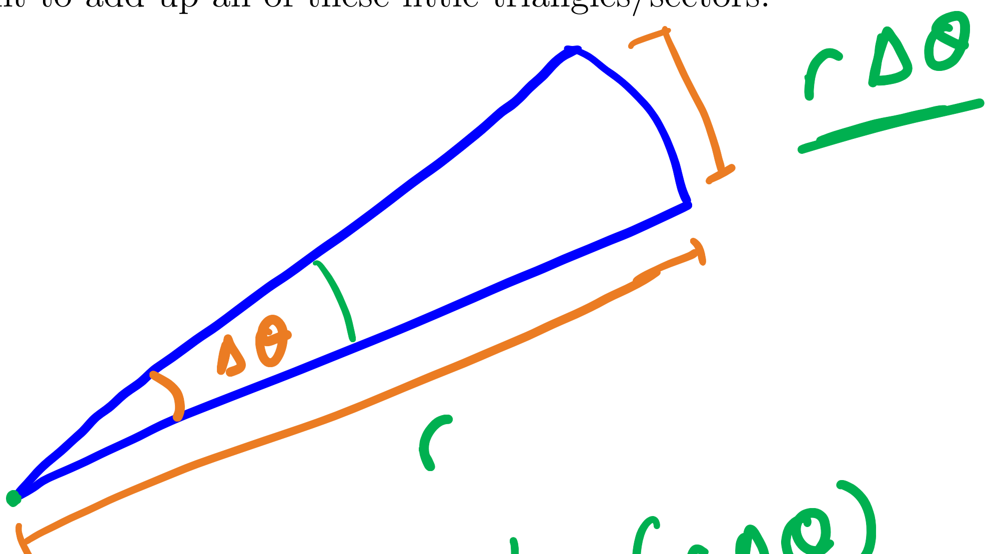
- Chop in little increments of θ



(A) Region $\alpha \le \theta \le \beta$ (B) Region divided into narrow sectors

Rogawski et al., *Calculus: Early Transcendentals*, 4e, © 2019 W. H. Freeman and Company

We want to add up all of these little triangles/sectors.



$$\text{Area} = \frac{1}{2} r (r\Delta\theta)$$

$$= \boxed{\frac{1}{2} r^2 \Delta\theta}$$

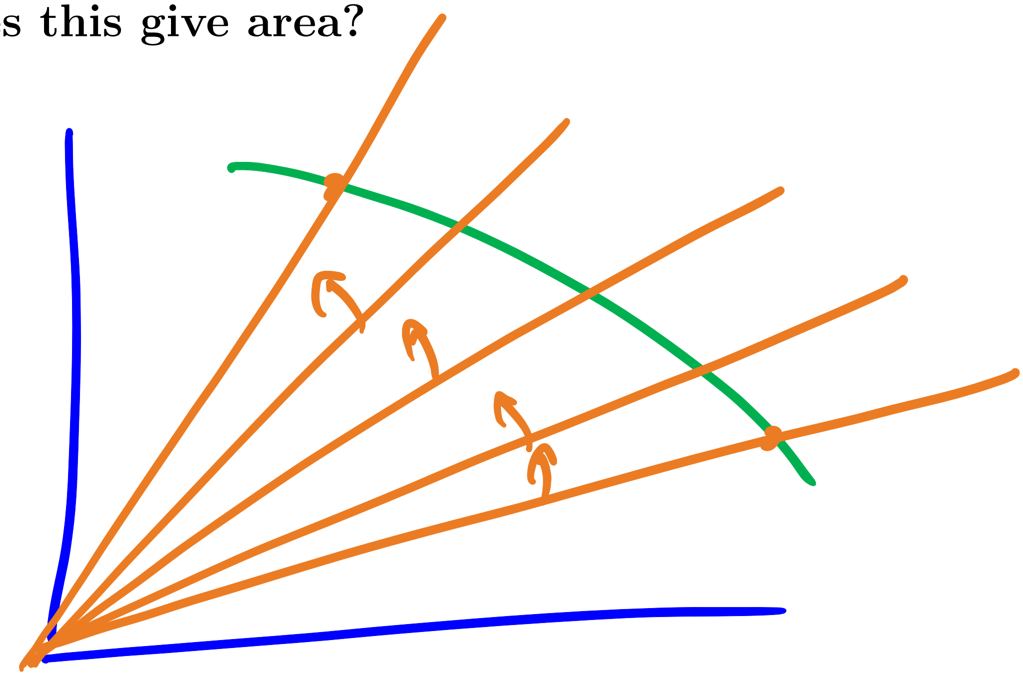
Theorem. *If f is a continuous function with $f \geq 0$ then the area bounded by a curve in polar form $r = f(\theta)$ and the rays $\theta = \alpha$ and $\theta = \beta$ is given by*

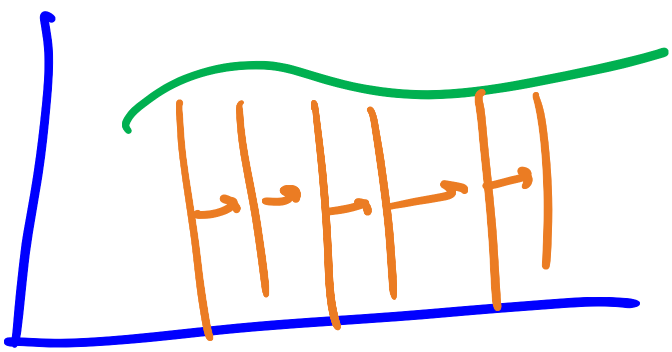
$$\boxed{\int_{\alpha}^{\beta} \frac{1}{2} r^2 \, d\theta = \int_{\alpha}^{\beta} \frac{1}{2} f(\theta)^2 \, d\theta}$$

Circle of radius R $\qquad f(\theta) = R$

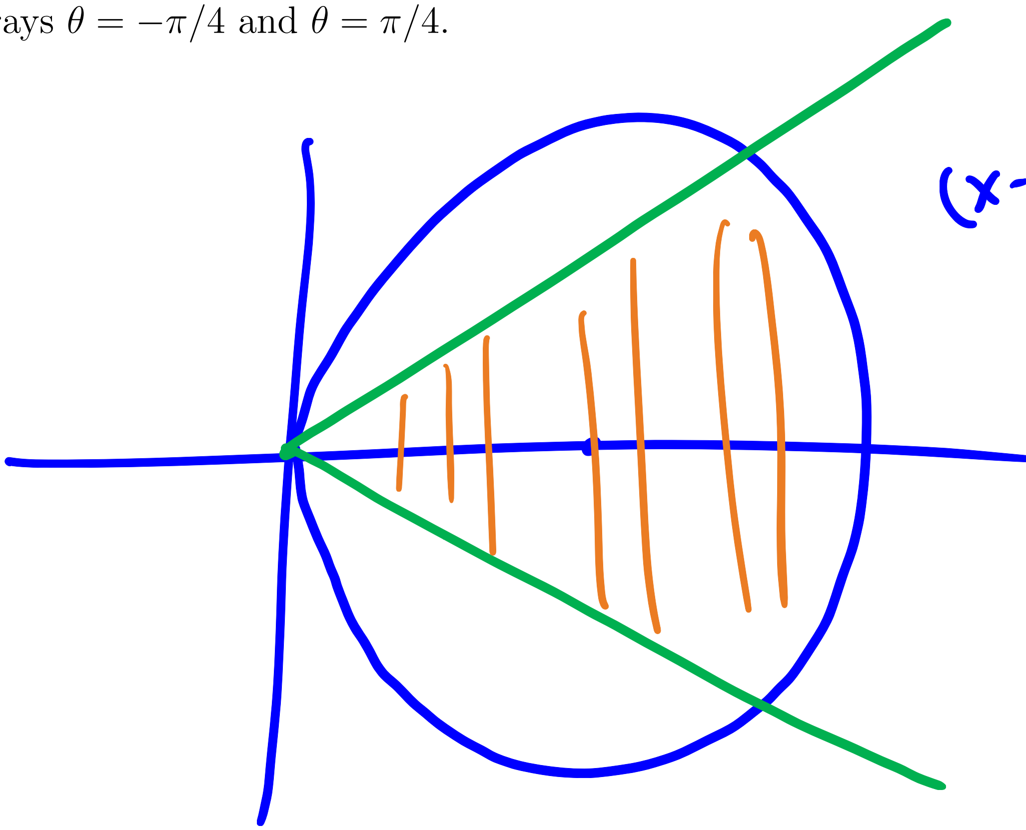
$$\int_0^{2\pi} \frac{1}{2} R^2 \, d\theta = \frac{1}{2} R^2 \theta \Big|_0^{2\pi} = \boxed{\pi R^2}$$

How does this give area?





Example: Find the area of the portion of the circle $r = 2\cos\theta$ between the rays $\theta = -\pi/4$ and $\theta = \pi/4$.



$(x-1)^2 + y^2 = 1$

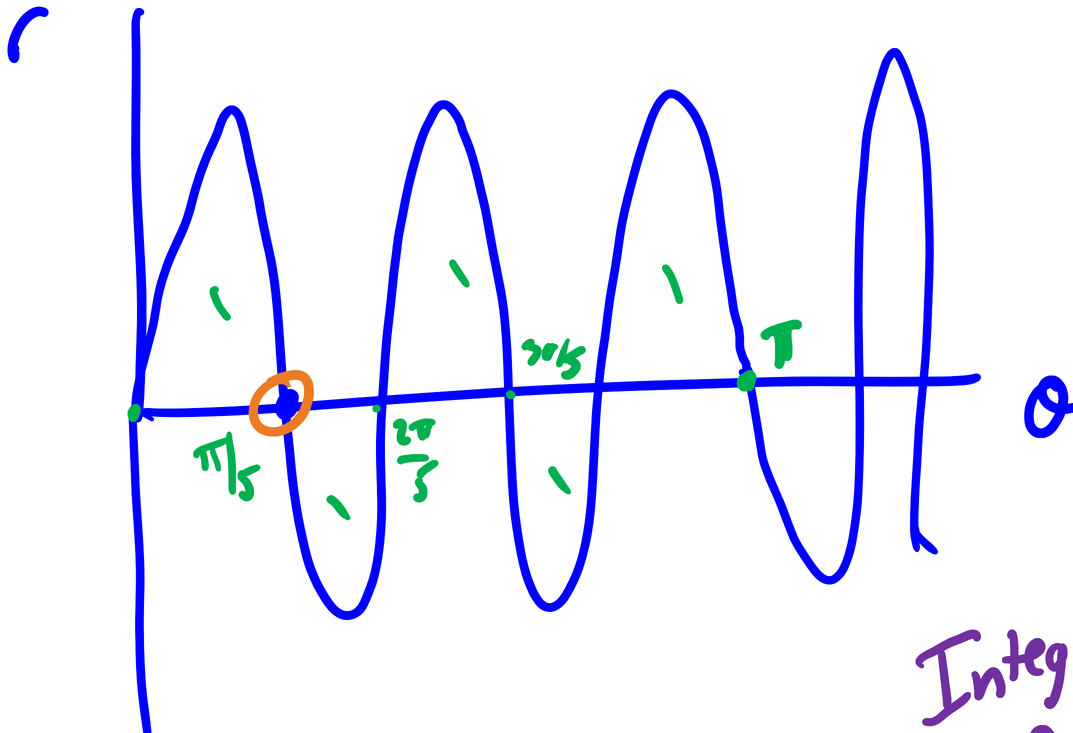
$$\int_{-\pi/4}^{\pi/4} \frac{1}{2}(f(\theta))^2 \, d\theta = \frac{1}{2}\int_{-\pi/4}^{\pi/4} 4\cos^2\theta \, d\theta$$

$$= \cancel{\tfrac{1}{2}} \cdot \cancel{4} \int_{-\pi/4}^{\pi/4} \frac{1+\cos 2\theta}{\cancel{2}} \, d\theta = \theta + \frac{\sin 2\theta}{2} \Big|_{-\pi/4}^{\pi/4}$$

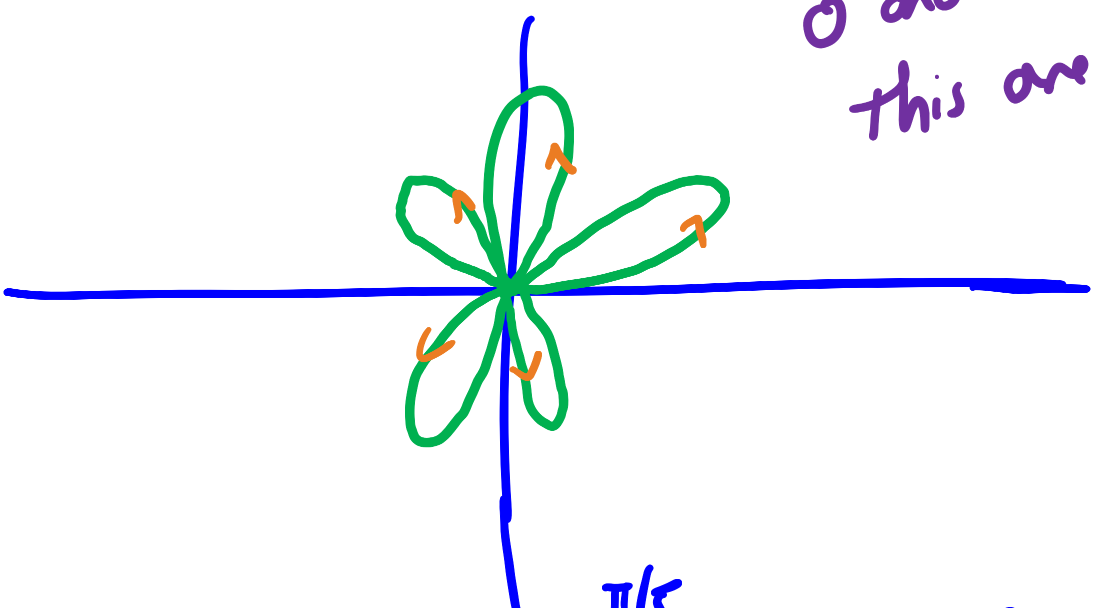
$$= \pi/4 + \frac{\cancel{\sin \pi/2}^{\,1}}{2} - \left(-\pi/4 + \frac{\cancel{\sin(-\pi/2)}^{\,-1}}{2}\right) = \boxed{\frac{\pi}{2} + 1}$$

2 More Examples

Example: Find the area of one petal of the graph $r = \sin 5\theta$



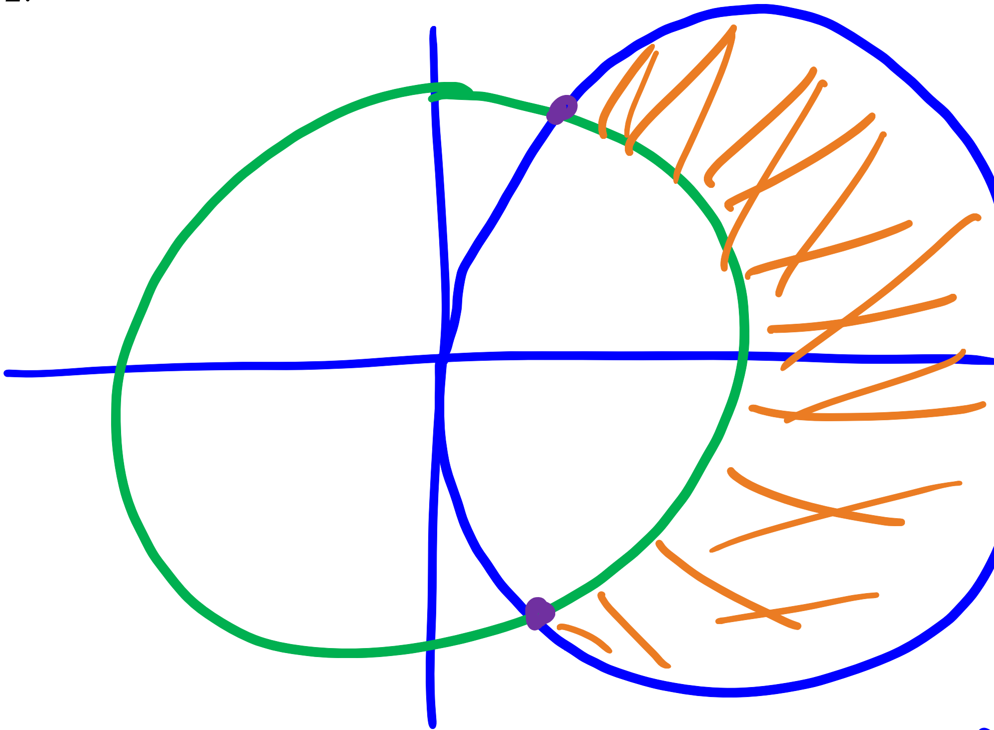
Integrate between 0 and π/5 for this area.



$$\int_0^{\pi/5} \frac{1}{2}\sin^2(5\theta)\,d\theta = \frac{1}{4}\int_0^{\pi/5} 1-\cos(10\theta)\,d\theta$$

$$= \frac{1}{4}\left(\theta - \frac{\sin(10\theta)}{10}\right)\Bigg|_0^{\pi/5} = \boxed{\pi/20}$$

Example: Find the area inside the circle $r = 4\cos\theta$ and outside the circle $r = 2$.



$4\cos\theta = 2$

$\cos\theta = 1/2$

$\theta = \pm \pi/3$

$$\text{Area} = \int_{-\pi/3}^{\pi/3} \frac{1}{2}(4\cos\theta)^2 - \frac{1}{2}(2)^2 \, d\theta$$

$$= \int_{-\pi/3}^{\pi/3} 8\underbrace{\cos^2\theta}_{\frac{1+\cos 2\theta}{2}} - 2 \, d\theta = \int_{-\pi/3}^{\pi/3} 2 + 4\cos 2\theta \, d\theta$$

$$= 2\theta + 2\sin 2\theta \Big|_{-\pi/3}^{\pi/3} = \boxed{\frac{4\pi}{3} + 2\sqrt{3}}$$

3 Arc Length

Let's now try to figure out a new arc length formula in polar coordinates. What was our parametric formula?

$$s = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$$

How can we get a formula in polar coordinates?

We have x and y in terms of r and θ.

→ Set $r = f(\theta)$

→ Plug everything in

→ Hope for a nice formula

$$\sqrt{x'(t)^2 + y'(t)^2}$$

$$x(\theta) = r\cos\theta = f(\theta)\cos\theta$$

$$x'(\theta) = f'(\theta)\cos\theta - f(\theta)\sin\theta$$

$$y(\theta) = f(\theta)\sin\theta$$

$$y'(\theta) = f'(\theta)\sin\theta + f(\theta)\cos\theta$$

$$x'(\theta)^2 = f'(\theta)^2\cos^2\theta \cancel{- 2f'(\theta)f(\theta)\cos\theta\sin\theta} + f(\theta)^2\sin^2\theta$$

$$+\, y'(\theta)^2 = f'(\theta)^2\sin^2\theta \cancel{+ 2f'(\theta)f(\theta)\cos\theta\sin\theta} + f(\theta)^2\cos^2\theta.$$

$$\boxed{x'(\theta)^2 + y'(\theta)^2 = f'(\theta)^2 + f(\theta)^2}$$

Theorem. *Let $f'(\theta)$ be continuous on $[\alpha, \beta]$. Then the arc length s of the curve $r = f(\theta)$ is given by*

$$\boxed{s = \int_{\alpha}^{\beta} \sqrt{\underbrace{f(\theta)^2 + f'(\theta)^2}_{x'(\theta)^2 + y'(\theta)^2}}\, d\theta}$$

→ Might be impossible to work out by hand.

Example: Find the length of one petal the curve $r = \sin(5\theta)$.

We know that one petal is from

$\theta = 0$ to $\theta = \pi/5$

$f(\theta) = \sin(5\theta)$ $\qquad$ $f'(\theta) = 5\cos(5\theta)$

$$s = \int_0^{\pi/5} \sqrt{\sin^2(5\theta) + 25\cos^2(5\theta)}\, d\theta$$

$$s = \int_0^{\pi/5} \sqrt{1 + 24\cos^2(5\theta)}\, d\theta$$

$$\boxed{s \approx 2.101}$$