Polar Coordinates

Learning Goals

- Locate points in a plane by using polar coordinates
- Convert coordinates from polar form to rectangular form and vice versa
- Convert a Cartesian equation to polar form and vice versa
- Graph polar equations by plotting points and find zeros and maximum values for a polar equation

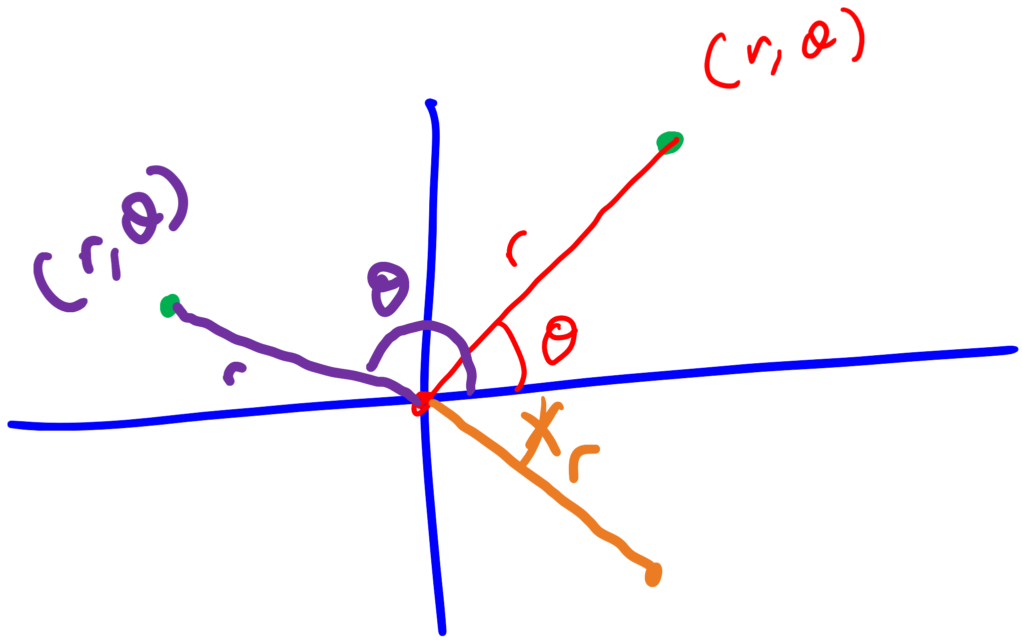
Contents

1 Polar Coordinates

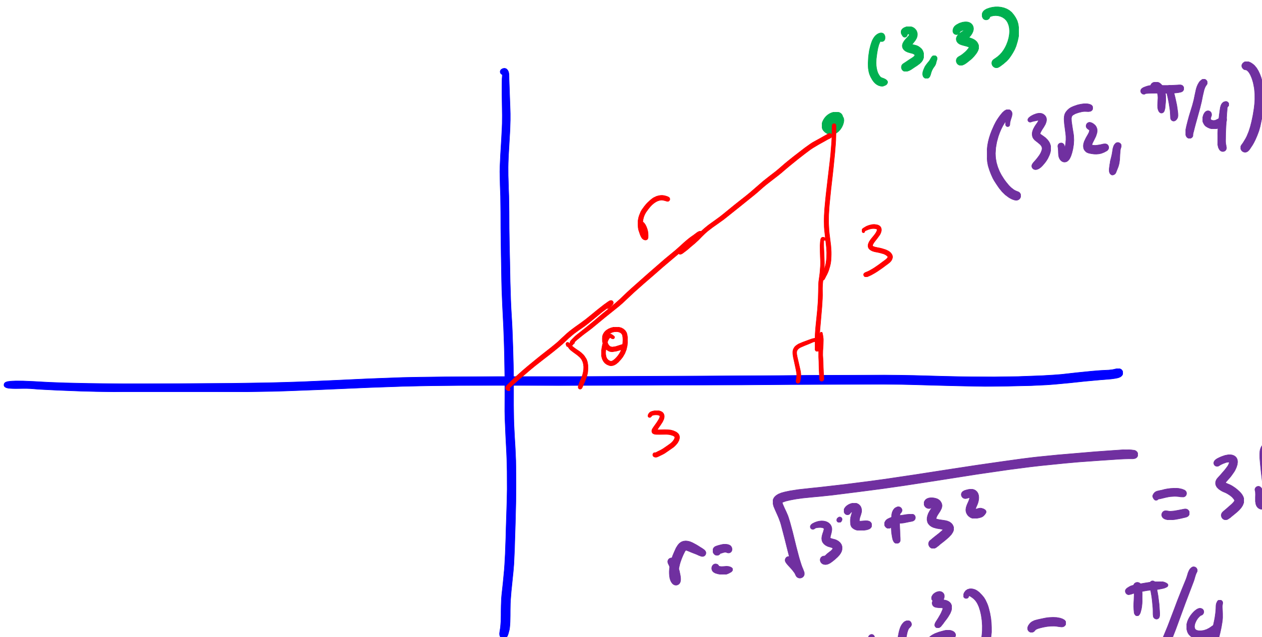
Polar coordinates give a new way to interpret equations or graphs that may make it easier to analyze. The new set of coordinates (r, θ) is defined as follows:

(x, y) - Cartesian

1. r is distance of the point from the origin
2. θ is the angle that the line from the point to the origin makes with the positive x axis in the counterclockwise direction.

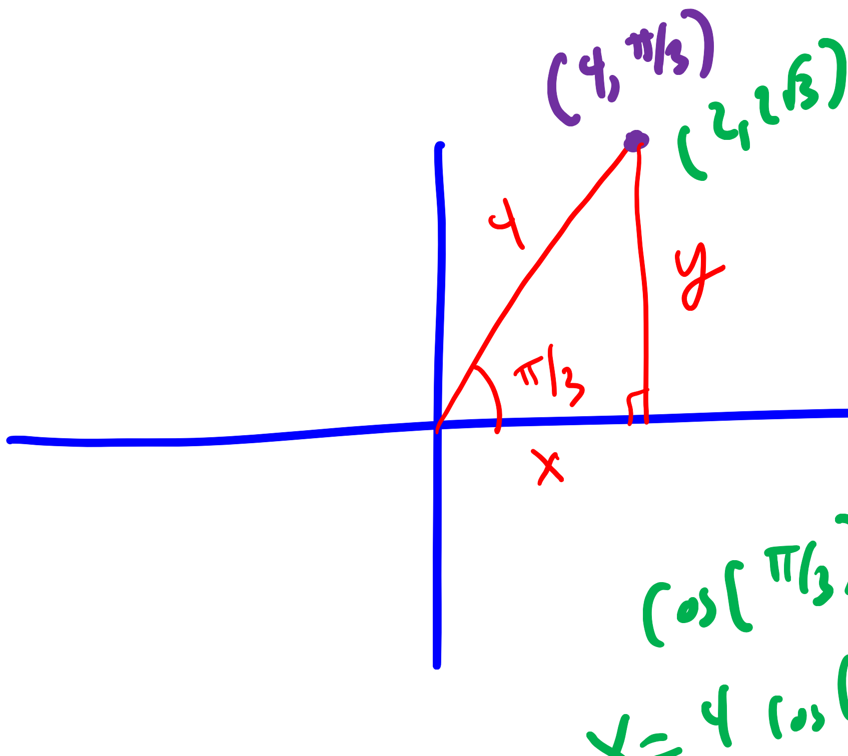


Example: What are the polar coordinates of the point $(x, y) = (3, 3)$? What are the rectangular coordinates of the point $(r, \theta) = (4, \pi/3)$?



$r = \sqrt{3^2 + 3^2} = 3\sqrt{2}$

$\theta = \tan^{-1}\left(\frac{3}{3}\right) = \pi/4$



$\sin(\pi/3) = \frac{y}{4}$

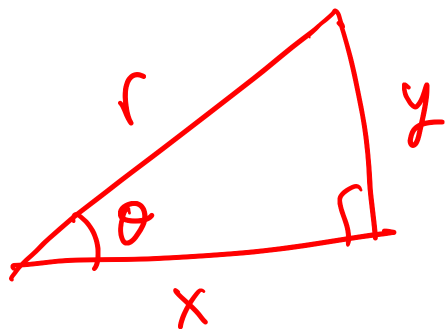
$y = 4\sin(\pi/3) = 2\sqrt{3}$

$\cos(\pi/3) = \frac{x}{4}$

$X = 4\cos(\pi/3) = 2$

2 Conversion Formulas

How do we get between the different coordinate systems?



$$r = \sqrt{x^2 + y^2}$$

$$x = r\cos\theta$$

$$\underbrace{\theta = \tan^{-1}(y/x)}$$

Need to make sure you land in the right quadrant.

$$y = r\sin\theta$$

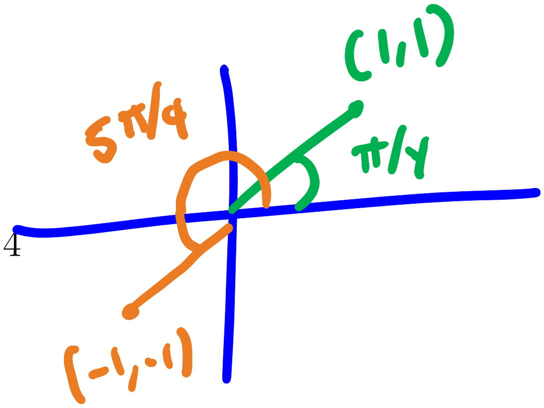
Cartesian Points

$(1,1)$ $(-1,-1)$

$$r = \sqrt{2}$$

$$\theta = \tan^{-1}\left(\frac{1}{1}\right) \text{ or } \tan^{-1}\left(\frac{-1}{-1}\right)$$

$$\tan^{-1}(1) = \pi/4$$



In general, we assume that r and θ can be any real numbers. This means that the expression of a given point in the plane is not unique.

Cartesian

$(1,1)$

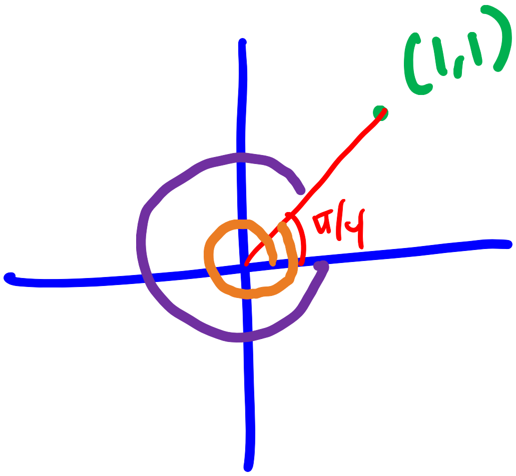
Polar

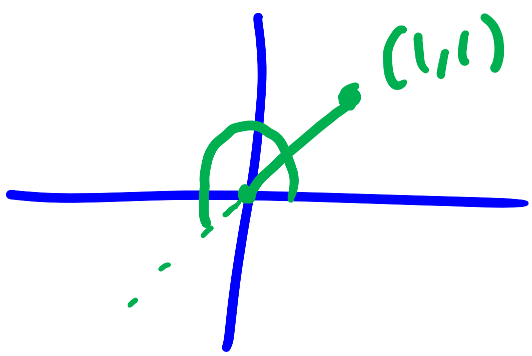
$(\sqrt{2}, \pi/4)$

$(\sqrt{2}, 9\pi/4)$

$(\sqrt{2}, -7\pi/4)$

$(-\sqrt{2}, 5\pi/4)$





To get unique: $r > 0, \ 0 \leq \theta < 2\pi$

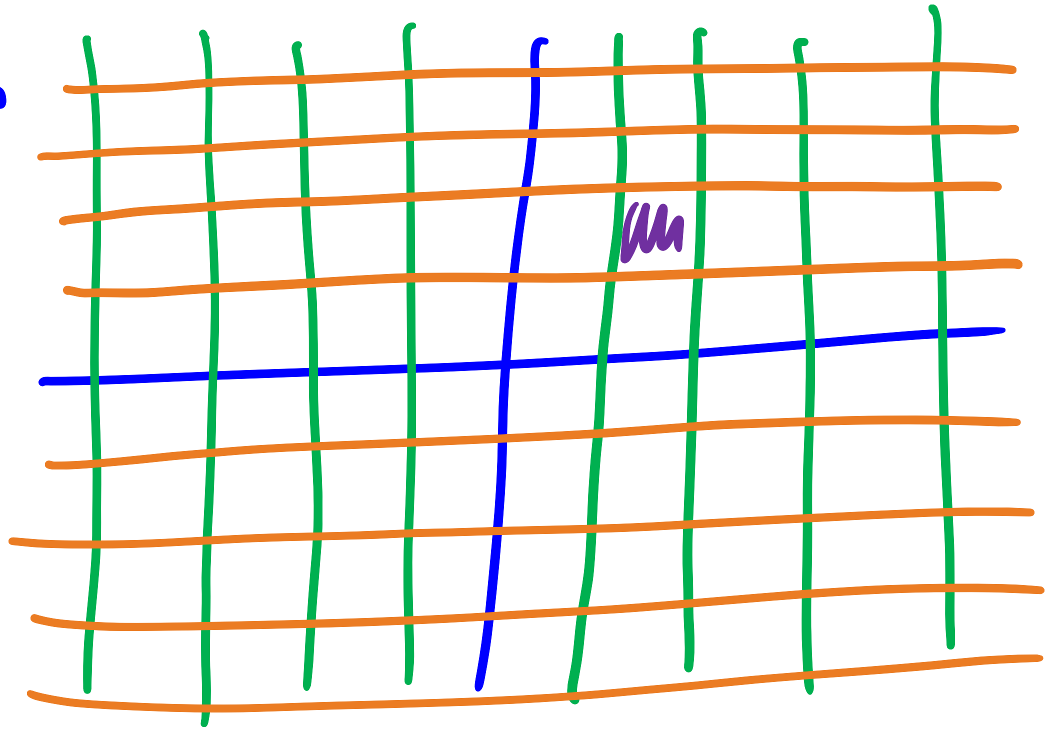
(no way to write the origin)

Comparing Coordinate Systems

To compare the systems, let's think about what happens in each system when one variable is held constant.

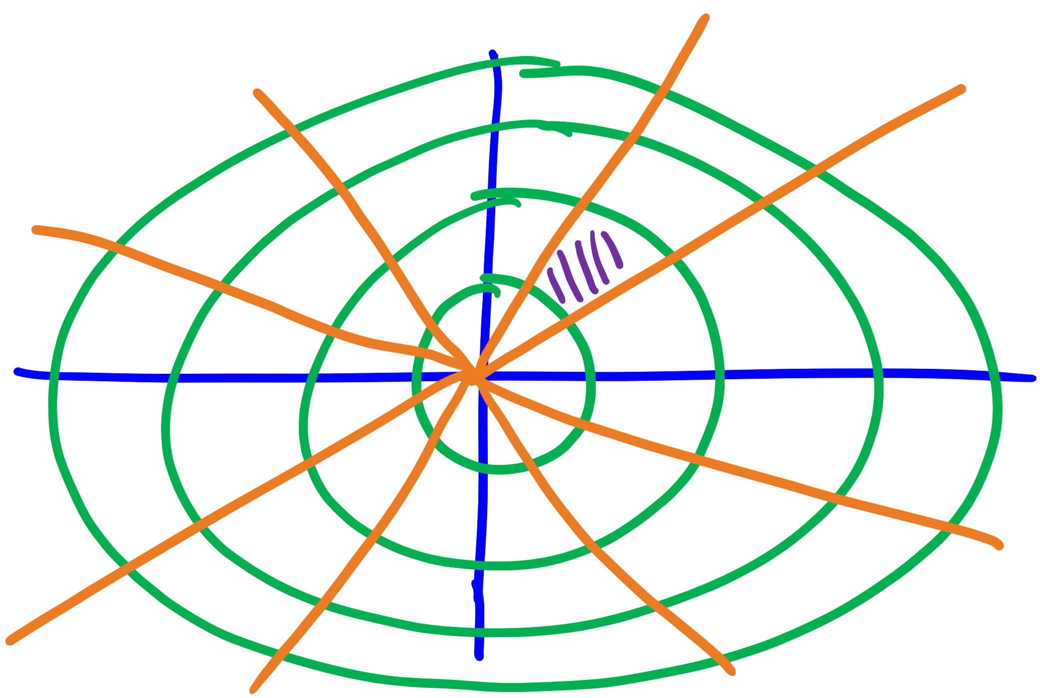
Cartesian

- Fix x
- Fix y

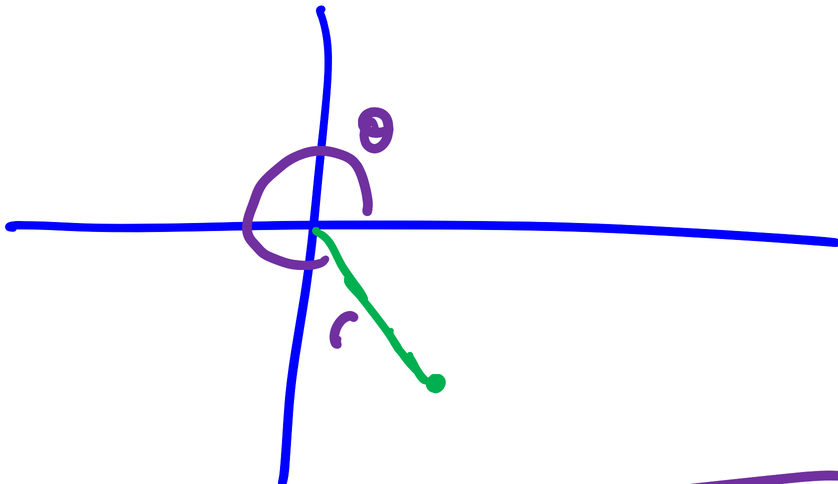


Polar

- Fix r
- Fix θ



Example: What is the polar coordinate representation of $(1, -3)$? Find at least 3 different ways to represent this point.



$$r = \sqrt{1^2 + 3^2} = \sqrt{10}$$

$$\theta = \tan^{-1}(-3) \quad (\text{between } -\pi/2 \text{ and } 0)$$

- $\left(\sqrt{10}, \tan^{-1}(-3)\right)$, $\boxed{\left(\sqrt{10}, \tan^{-1}(-3) + 2\pi\right)}$
- $\left(-\sqrt{10}, \tan^{-1}(-3) + \pi\right)$

3 Polar Equations

$y = f(x)$

When we want to describe curves in polar coordinates, we generally try to do so in the form $r = f(\theta)$. To do this, we can try to use the ideas of polar coordinates directly, or use our conversion formulas to convert an equation involving x and y, to one in terms of r and θ.

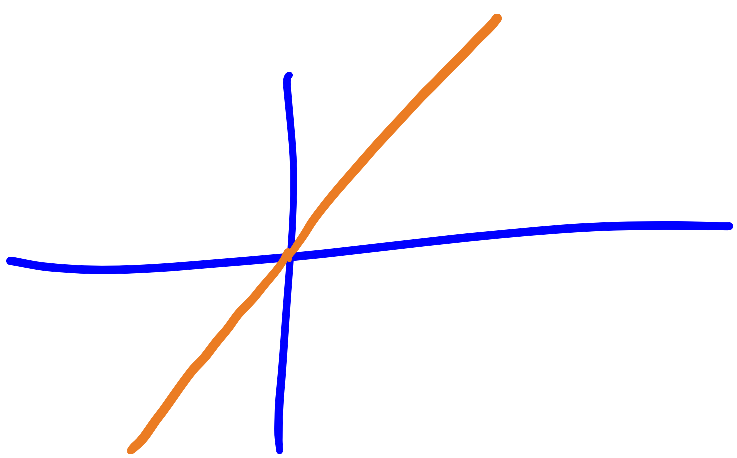
Example: Find the equation of the line $y = 2x$ in polar coordinates.

$y = r\sin\theta \qquad x = r\cos\theta$

$$\cancel{r}\sin\theta = 2\cancel{r}\cos\theta$$

$$\frac{\sin\theta}{\cos\theta} = 2 \qquad \tan\theta = 2$$

$$\theta = \tan^{-1}(2)$$



Fits a curve where θ is constant.

Example: Find the equation of the line $y = 3 - 4x$ in polar coordinates.

$$r\sin\theta = 3 - 4r\cos\theta$$

$$r\sin\theta + 4r\cos\theta = 3$$

$$\boxed{r = \frac{3}{\sin\theta + 4\cos\theta}}$$

Straight line

$$\boxed{r = d\sec(\theta - \alpha)}$$

where the closest point on the line to the origin is at polar coordinates $\boxed{(d, \alpha)}$

4 Converting Equations from Polar

There are a few things to keep in mind when converting equations from polar to Cartesian variables.

- Look for good terms in your expression

→ $r\sin\theta$ $\quad r^2 \quad \tan\theta$

$r\cos\theta$

→ Manipulate to get as many of these good terms as possible before you convert.

Example: Find the rectangular equation corresponding to the polar equation $r = 4\sin\theta$.

multiply by r

$$\underbrace{r^2}_{} = 4\underbrace{r\sin\theta}_{}$$

$$x^2+y^2 = 4y$$

$$x^2+y^2-4y+4 = 0+4$$

$$\boxed{x^2+(y-2)^2 = 4}$$

$r = 4\sin\theta$

- Circle
- Radius 2
- Center $(0,2)$

5 Graph Sketching

Polar graphs can be sketched in the same way as rectangular ones; plotting points and connecting them. The plotting part just needs to be interpreted in the correct way.

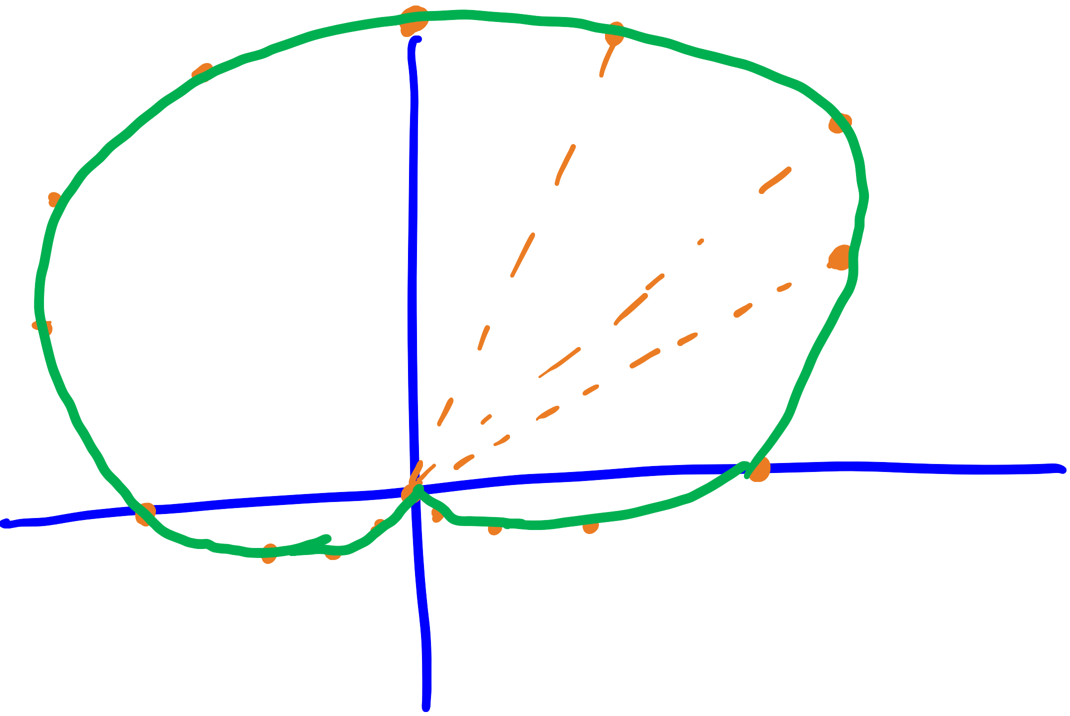
θ = independent variable

r = dependent variable

$$r = f(\theta)$$

- Plug in θ
- Plot as (r, θ) points, not (x, y)

Example: Sketch the graph of $r = 1 + \sin\theta$.



θ	r
0	1
$\pi/6$	$3/2$
$\pi/4$	$1+\sqrt{2}/2$
$\pi/3$	$1+\sqrt{3}/2$
$\pi/2$	2
$2\pi/3$	$1+\sqrt{3}/2$
π	1
$7\pi/6$	$1/2$
$5\pi/4$	$1-\sqrt{4}/2$
$4\pi/3$	$1-\sqrt{3}/2$
$3\pi/2$	0