Arc Length and Speed for Parametric Equations

Learning Goals

- Find speed of a particle moving on a parametric curve
- Find the arc length of a curve defined by parametric equations
- Find the surface area of a volume of revolution generated by revolving a parametrically defined curve about the x-axis or y-axis

Contents

1 Arc Length for Parametric Curves

Now, we want to deal with arc length, speed, and surface area for parametric curves. How can we figure out how long curves are when expressed this way?

Recall: Formulas for arc length and surface area for functions.

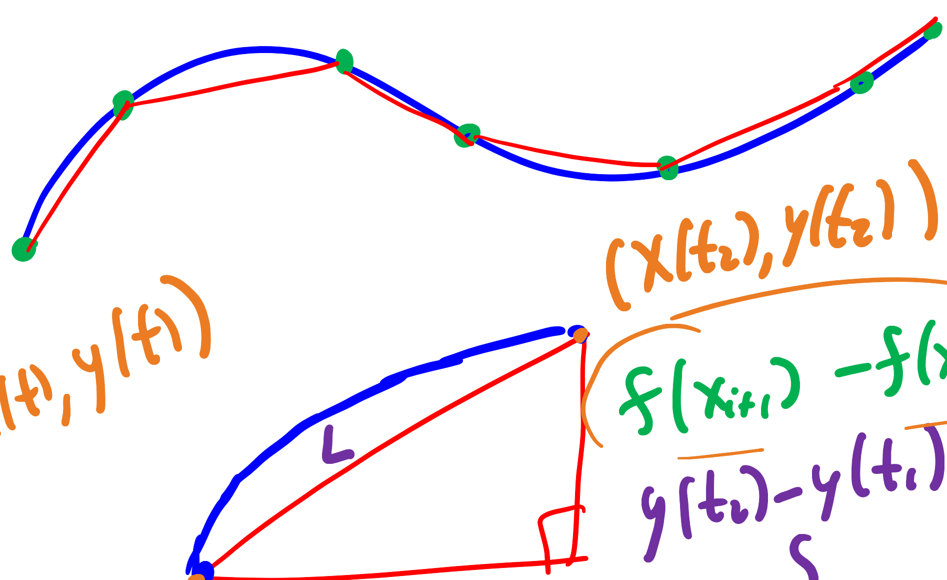
$$s = \int_a^b \sqrt{1 + (f'(x))^2}\, dx$$

$$S = 2\pi \int_a^b f(x) \sqrt{1 + (f'(x))^2}\, dx$$

How did we calculate arc length before?

- Breaking the curve up into line segments.
- Use Pythagorean Theorem to find the length.

What does this look like for these equations?



$C(t) = (x(t), y(t))$

$(x(t_2), y(t_2))$

$f(x_{i+1}) - f(x_i)$

$y(t_2) - y(t_1) \approx y'(d)\,\Delta t$

$(x(t_1), y(t_1))$

$x_{i+1} - x_i$

$x(t_2) - x(t_1) \approx x'(c)\,\Delta t$

$$L = \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

$$L = \sqrt{(x'(c)\,\Delta t)^2 + (y'(d)\,\Delta t)^2}$$

$$L = \Delta t \sqrt{(x'(c))^2 + (y'(d))^2}$$

$$\int \sqrt{x'(t)^2 + y'(t)^2}\ dt$$

Theorem. *Let $c(t) = (x(t), y(t))$ be a parametrization that directly traverses C for $a \leq t \leq b$. Assume that $x'(t)$ and $y'(t)$ exist and are continuous. Then the arc length s of C is equal to*

$$s = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$$

What do we mean by **directly traverses**? Why is this important?

→ Only pass over every point once.

$c(\theta) = (\cos\theta, \sin\theta)$ $\quad 2\pi$

$\theta = 0$ to $\theta = 6\pi$ → Arc Length of 6π

→ Don't want to "double count" parts of this curve → Gone around 3 times

Example: Find the length of the curve $x(t) = 3t^2$, $y(t) = 4t^3$ from $1 \le t \le 4$.

$$s = \int_1^4 \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$$

$$s = \int_1^4 \sqrt{(6t)^2 + (12t^2)^2}\, dt$$

$$s = \int_1^4 \sqrt{36t^2 + 144t^4}\, dt$$

$$s = \int_1^4 6t\sqrt{1+4t^2}\, dt$$

$u = 1+4t^2$

$du = 8t\, dt$

$$s = \frac{6}{8}\int_5^{65} \sqrt{u}\, du = \frac{3}{4}\cdot\frac{2}{3}u^{3/2}\Big|_5^{65}$$

$$= \boxed{\frac{1}{2}\left(65^{3/2} - 5^{3/2}\right)}$$

2 Distance Travelled vs. Displacement

What is the difference between these two things?

Displacement: End − Start, straight line between them.

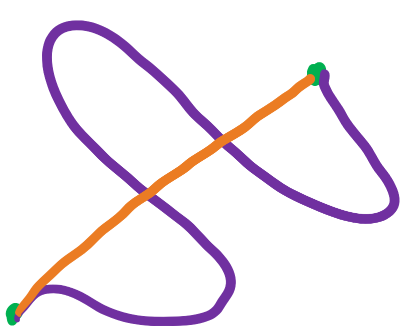
- Doesn't care how you get there.

Distance Traveled: How many steps did I take to get there?

- Route matters.

speed: $\sqrt{(x'(t))^2+(y'(t))^2}$

* Integrating speed gives distance traveled *



Displacement: Find start and end.

Example: Consider the path $c(t) = (t^2, e^t)$. What is the speed at $t = 1$? What is the distance travelled and displacement between $t = 1$ and $t = 3$?

Speed: $\sqrt{(x'(t))^2 + (y'(t))^2}$

$= \sqrt{(2t)^2 + (e^t)^2}$

at $t=1$ Speed $\boxed{\sqrt{4+e^2}}$

Distance Traveled: $\int_1^4 \sqrt{4t^2 + e^{2t}}\,dt$

≈ 19.225

Displacement: $c(3) = (9, e^3)$ $c(1) = (1, e)$

Disp $= \sqrt{(9-1)^2 + (e^3-e)^2} = \sqrt{64 + (e^3-e)^2}$

≈ 19.121

3 Surface Area

We did surface area before by adding up the boundary area of little cylinders. We can do the same thing here.

Theorem. *Let $c(t) = (x(t), y(t))$, where $y(t) \geq 0$, $x(t)$ is increasing, and $x'(t), y'(t)$ are continuous. Then the surface obtained by rotating $c(t)$ about the x axis for $a \leq t \leq b$ has surface area*

$$S = \int_a^b 2\pi\, y(t) \sqrt{x'(t)^2 + y'(t)^2}\, dt$$

$$2\pi \quad r \quad h$$

$$2\pi \quad f(x) \sqrt{1 + f'(x)^2}$$

Example: Find the surface area of the surface generated by revolving $c(t) = (\cos^3(t), \sin^3(t))$ for $0 \le t \le \frac{\pi}{2}$.

$$S = \int_0^{\pi/2} 2\pi \underbrace{\sin^3(t)}^{y(t)} \sqrt{(x'(t))^2 + (y'(t))^2}$$

$$x'(t) = -3\cos^2(t)\sin(t)$$

$$y'(t) = 3\sin^2(t)\cos(t)$$

$$S = \int_0^{\pi/2} 2\pi \sin^3(t) \sqrt{9\cos^4(t)\sin^2(t) + 9\sin^4(t)\cos^2(t)}\, dt$$

$$S = \int_0^{\pi/2} 2\pi \sin^3(t) \cdot \cos(t)\sin(t) \cdot 3\sqrt{\cos^2(t) + \sin^2(t)}\, dt$$

$$S = \int_0^{\pi/2} 6\pi \sin^4(t)\cos(t)\,dt$$

$u = \sin(t) \qquad du = \cos(t)\,dt$

$$S = 6\pi \int_0^1 u^4\,du$$

$$= 6\pi \left.\frac{u^5}{5}\right|_0^1 = \boxed{\frac{6\pi}{5}}$$