Parametric Equations

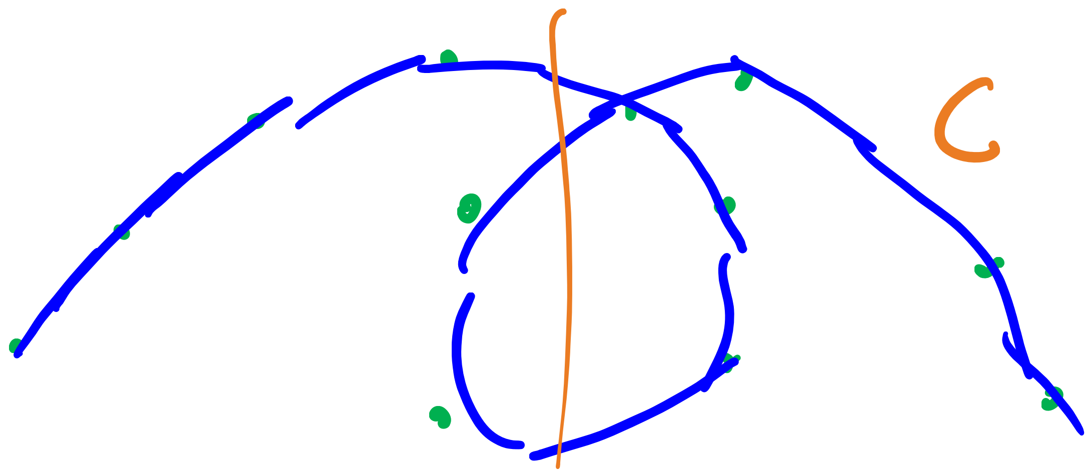
Learning Goals

- Parameterize a curve
- Find the parametric equations for a line segment given an orientation
- Eliminate the parameter in linear, polynomial, radical, exponential, logarithmic, or trigonometric equations
- Graph parametric equations by plotting points
- Find the derivative of a curve defined by parametric equations
- Find the equation of a line tangent to a parametrically defined curve

Contents

1 Definition of Parametric Equations

Imagine a particle moving along a curve C in the plane. To express this mathematically, we want to write the position of the particle as a function of time, and we will do this with two functions $x(t)$ and $y(t)$ for the x and y coordinate respectively.



$$c(t) = (x(t), y(t))$$

x coordinate as a function of time

y coordinate as a function of time.

c(t) is a parametrization of C with parametr t.

C is a parametrized curve.

We can also write this as **parametric equations**

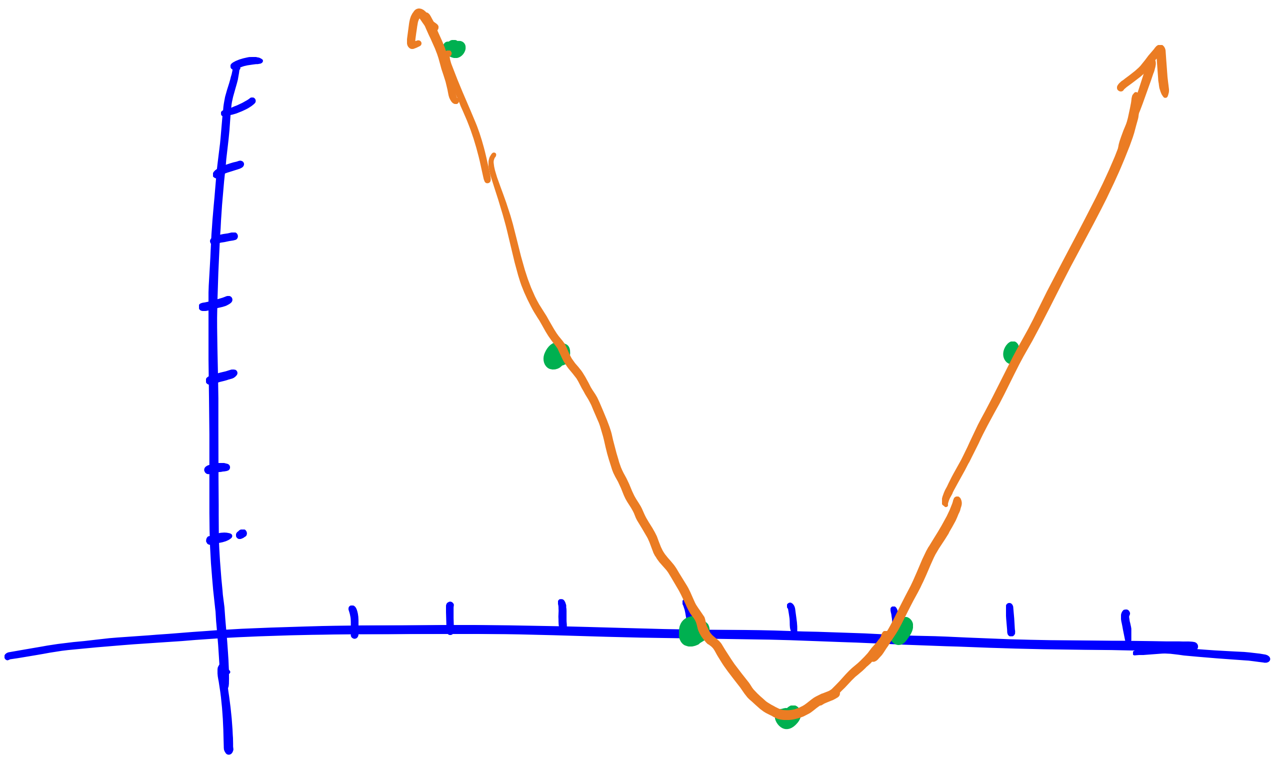
$$x = t^2 + 3t$$

$$y = \sin(t)$$

Instead of $\quad c(t) = (t^2 + 3t, \sin(t))$

Example: Sketch the curve given parametrically by

$$x = 4 - t \qquad y = t^2 + 2t$$



t	$x = 4 - t$	$y = t^2 + 2t$
0	4	0
1	3	3
2	2	8
-1	5	-1
-2	6	0
-3	7	3

2 Eliminating the Parameter

When you want to sketch out the graph of a parametric curve, or figure out how to deal with these functions, the easiest way to do it is by trying to eliminate the parameter.

- Get rid of the t, and just be left with x and y

→ Generally know what these look like.

Example: Figure out the relation between x and y for the parametric equation

$$\underbrace{x = 4 - t}, \quad y = t^2 + 2t$$

$t = 4 - x$

$y = (4-x)^2 + 2(4-x) = x^2 - 10x + 24$

$= 16 - 8x + x^2 + 8 - 2x$

$y = (x-6)(x-4)$

- Parabola
- upward
- zeros at $x=6$, $x=4$

Another way this can be done is by trying to find a relation between x and y based on the equations given for them.

Example: Figure out the relation between x and y for the parametric equation

$$x = 2 + 3\cos\theta \qquad y = 1 + 3\sin\theta$$

Know: $\cos^2\theta + \sin^2\theta = 1$

$x = 2 + 3\cos\theta \rightarrow \cos\theta = \frac{x-2}{3}$

$y = 1 + 3\sin\theta \rightarrow \sin\theta = \frac{y-1}{3}$

$$\left(\frac{x-2}{3}\right)^2 + \left(\frac{y-1}{3}\right)^2 = 1$$

$$(x-2)^2 + (y-1)^2 = 9$$

- Circle
- Radius 3
- Center $(2,1)$

3 Multiple Parametrizations

For any curve C, there are many ways to write a function $c(t)$ so that the particle moves along the curve. Sometimes, we refer to $c(t)$ as a *path*, which indicates that it's not just the curve C but also the way the particle moves along the path.

$x = 2 + 3\cos\theta \qquad y = 1 + 3\sin\theta$

$x = 2 + 3\cos(\underline{2\theta}) \qquad y = 1 + 3\sin(\underline{2\theta})$

$x = 2 + 3\cos(-\theta) \qquad y = 1 + 3\sin(-\theta)$

$x = 2 + 3\cos(\theta - \pi/4) \qquad y = 1 + 3\sin(\theta - \pi/4)$

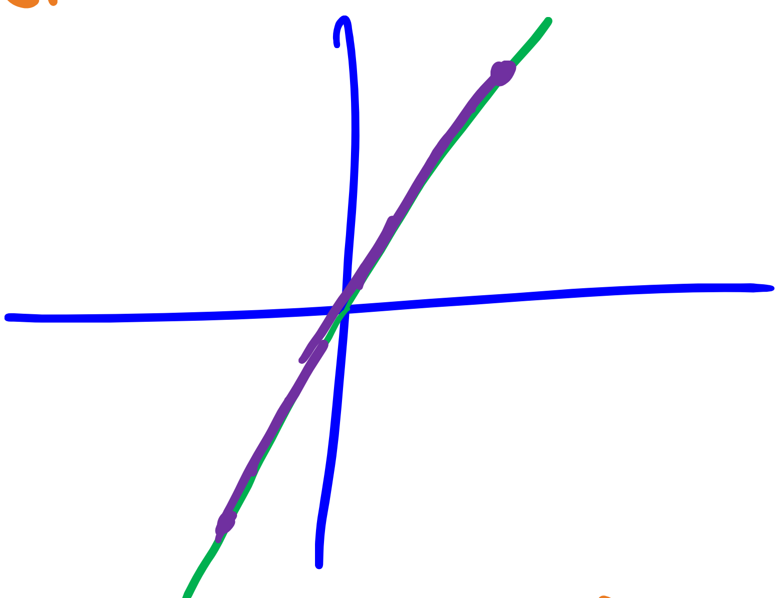
→ All of these trace out the same circle.

Example: Consider the following three parametrizations. What curve do they trace out? How do they move along this curve?

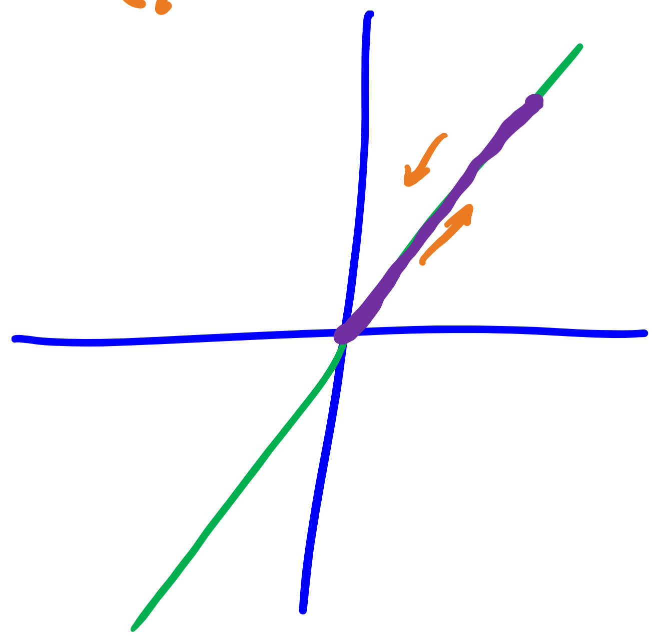
$$c_1(t) = (t, 2t) \qquad c_2(t) = (t^2, 2t^2) \qquad c_3(t) = (\sin(t), 2\sin(t))$$

$\rightarrow \; y = 2x$

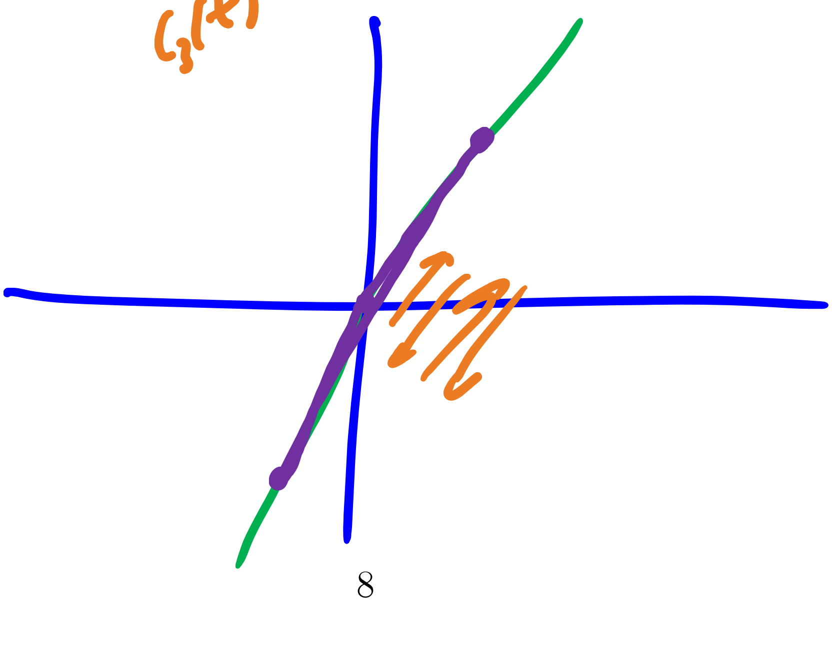
$c_1(t)$



$c_2(t)$



$c_3(t)$



4 Tangent Lines to Parametric Curves

With a curve like this, what is the slope of the tangent line?

$$c(t) = (x(t), y(t))$$

Tangent Line

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

→ Come from slopes of secant lines

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

$$y(t_1) - y(t_0)$$

$$\Delta y = y'(c)\,\Delta t$$

Mean Value Theorem

$$\Delta x = x'(d)\,\Delta t$$

$$\lim_{\Delta t \to 0} \frac{y'(c)\cancel{\Delta t}}{x'(d)\cancel{\Delta t}} = \boxed{\frac{y'(t)}{x'(t)}}$$

Derivative

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

To find slope of tangent line,

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)}$$

→ Can use $x(t), y(t)$ as a point to then find the line.

Example: Consider the parametric curve $c(t) = (t^2 - 9, 8t - t^3)$. Find an expression (in terms of t) for $\frac{dy}{dx}$. When is the tangent line horizontal? When is it vertical?

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{8-3t^2}{2t}$$

Horizontal? $\frac{dy}{dx} = 0$

$8 = 3t^2$ or $\boxed{t \pm \sqrt{8/3}}$

Vertical? $\frac{dy}{dx}$ is undefined

$\frac{dx}{dy} = 0$

$2t = 0$

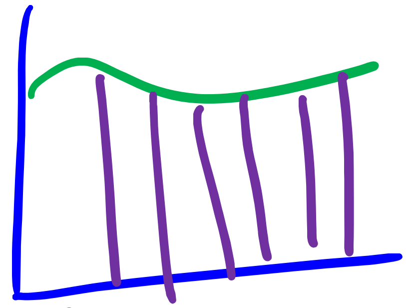
$t = 0$

$(-9, 0)$

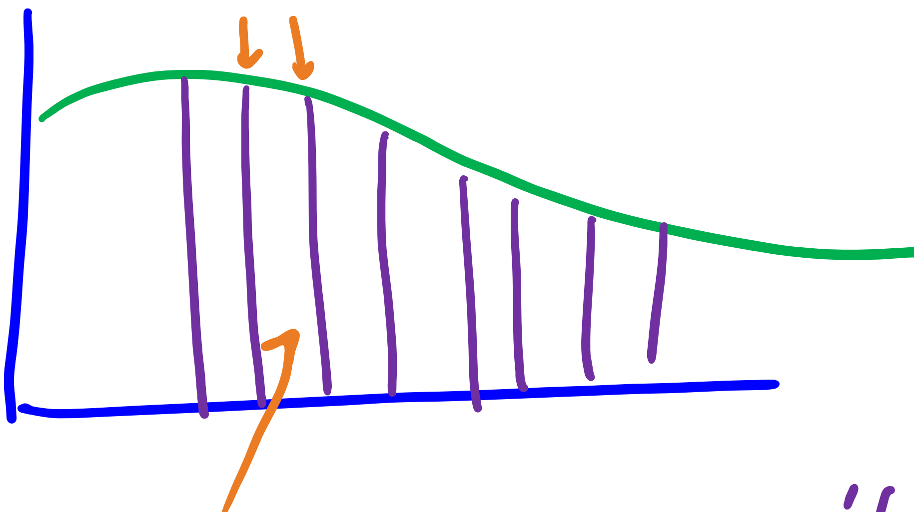
5 Area under a parametric curve

If we have $y = f(x)$, we know how to find the area between the graph and the x-axis.

$$\int_a^b f(x)\,dx$$



How can we do this if we have a parametric curve instead?



Area $\quad f(x)\,\Delta x \sim y(t)\,x'(t)\,\Delta t$

$x'(t)\,\Delta t$

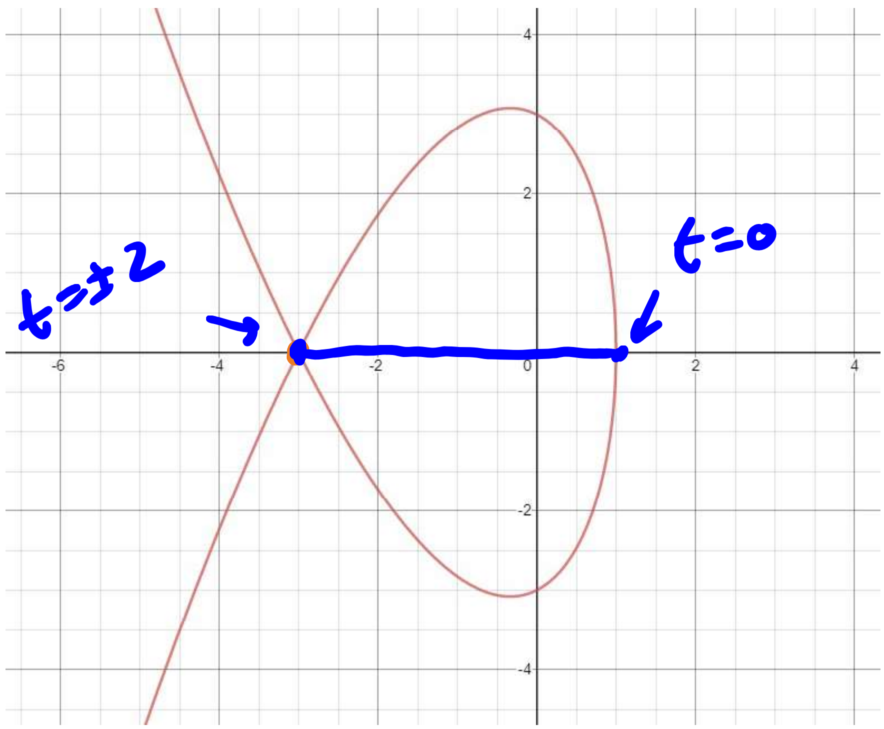
Height $y(t)$ $\quad$ width $\Delta x \sim x'\,\Delta t$

If $c(t) = (x(t), y(t))$ is always above the x-axis and $x(t)$ is increasing from $t=a$ to $t=b$ then the area between the graph and the x-axis is

$$\boxed{\int_a^b y(t)\, x'(t)\, dt.}$$

Example: Find the area contained inside the loop of the graph

$$c(t) = (1 - t^2, t^3 - 4t).$$



$y(t) = 0$ $\qquad$ $t^3 - 4t = 0$ $\qquad$ $2, -2, 0$

$t(t^2 - 4) = 0$

$$\int_0^2 (t^3 - 4t)(-2t)\,dt = \int_0^2 -2t^4 + 8t^2\,dt$$

$$-\frac{2}{5}t^5 + \frac{8}{3}t^3 \Big|_0^2 = -\frac{64}{5} + \frac{64}{3} = \frac{128}{15}$$

Area $= \boxed{\frac{256}{15}}$