Taylor Series

Learning Goals

- Recognize a Taylor series
- Represent a function at a value with a Taylor series and determine the interval of convergence
- Find the Maclaurin series for a function and show that the series converges
- Find the Maclaurin series for a trigonometric, logarithmic, or exponential function
- Find a Maclaurin series by differentiating another series

Contents

1 Definition of Taylor Series

Now we want to expand the idea of power series slightly. If we have a function $f(x)$, I want to be able to figure out what it's power series expansion should be. For certain types of functions, there were tricks, but what about for any function?

Let's think about this in the other direction first. Assume that we know that

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

Is there a way we can figure out what the a_0 needs to be?

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

$$\boxed{f(0) = a_0}$$

What about the other a_n coefficients?

$$f(x) = a_0 + \underbrace{a_1}\, x + a_2 x^2 + a_3 x^3 + \cdots$$

$$f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \cdots$$

$$\boxed{a_1 = f'(0)}$$

$$f''(x) = \underbrace{2}a_2 + \underbrace{6}a_3 x + 12 a_4 x^2 + \cdots$$

$$a_2 = \frac{f''(0)}{2!} \qquad a_3 = \frac{f'''(0)}{6} = \frac{f'''(0)}{3!}$$

$$\boxed{a_k = \frac{f^{(k)}(0)}{k!}}$$

This idea leads to our definition of a Taylor Series.

Definition. If f is infinitely differentiable at c, the **Taylor Series** of a f at c is the power series

$$T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$$

$$\frac{f^{(n)}(c)}{n!}$$

Maclaurin Series : Center $c = 0$

$$\boxed{M(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n}$$

Why is this important?

The main benefit here is that power series can be differentiated and integrated term by term, and the Taylor Series here is a power series.

- Can Integrate and differentiate Taylor Series Term by Term.

→ Can Truncate the series to get a polynomial.

$$\int e^{x^2}\,dx \qquad \int \sin(1/x)\,dx$$

☆ Requires that the Taylor Series Converges to f. ☆

2 Convergence of Taylor Series

The question to be dealt with now is when does the Taylor Series of a function actually converge to the function itself. It doesn't have to, but when it does, we get a lot of benefits from it, like the ability to integrate term by term. We'll talk about two ways to know this in this section.

Theorem.

If $f(x)$ has a power series representation on some interval I

→ This power series converges to f on I.

then this power series is also the Taylor series for f, and it converges to f on the same interval I.

→ Our Power series examples from before are also Taylor series!

Example: Verify that the Taylor Series for $\frac{1}{1-x}$ is $\sum_{n=0}^{\infty} x^n$, and state where this series is valid.

$f(x) = \frac{1}{1-x}$ has power series expansion $\sum_{n=0}^{\infty} x^n$

and is valid on $(-1, 1)$.

$f(x) = \frac{1}{1-x}$ $\qquad f(0) = \frac{1}{1} = 1 \qquad a_0 = \frac{1}{0!} = 1$

$f'(x) = (1-x)^{-2} \qquad f'(0) = 1 \qquad a_1 = \frac{1}{1!} = 1$

$f''(x) = (-2)(1-x)^{-3}(-1) \qquad f''(0) = 2 \qquad a_2 = \frac{2}{2!} = 1$

$f'''(x) = (-3)\left(2(1-x)^{-4}\right)(-1) \qquad f'''(0) = 3 \cdot 2 \qquad a_3 = \frac{3 \cdot 2}{3!} = 1$

$f^{(k)}(0) = k! \qquad a_k = \frac{f^{(k)}(0)}{k!} = 1$ for all k.

The Taylor Series for

$$\boxed{f(x) = \frac{1}{1-x} \quad \text{is} \quad \sum_{n=0}^{\infty} x^n}$$

Example: Verify that the Taylor Series for e^x is $\sum_{n=0}^{\infty} \frac{x^n}{n!}$.

$f(x) = e^x$ $\qquad$ $f(0) = 1$

$f'(x) = e^x$ $\qquad$ $f'(0) = 1$

$\vdots$ $\qquad$ $\vdots$

$f^{(k)}(x) = e^x$ $\qquad$ $f^{(k)}(0) = 1$

Coefficient $a_k = \dfrac{f^{(k)}(0)}{k!} = \dfrac{1}{k!}$

$\rightarrow$ Argument as to why this converges in the next video.

3 Proving Convergence

Next, we have a theorem that gives us convergence of Taylor Series. A proof will not be given here, but we can use it to do get some nice results.

Theorem. *Let $I = (c-R, c+R)$, where $R > 0$ and assume that f is infinitely differentiable on I. Suppose there exists $K > 0$ such that all derivatives of f are bounded by K on I, that is*

$$|f^{(k)}(x)| \leq K$$

for all $k \geq 0$ and all x in I. Then f is represented by its Taylor Series in I:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$$

- All bounded by the same K.

  → Strong condition.

What functions can we use this on? What functions do we know that have bounded derivatives?

Sine and Cosine! $K = 1$

Example: Find the Maclaurin series for $\sin x$.

$f(x) = \sin(x)$

$f(0) = 0$

$f'(x) = \cos(x)$ $\quad f'(0) = 1$ $\quad 0, 1, 0, -1, \ldots$

$f''(x) = -\sin(x)$ $\quad f''(0) = 0$

$f'''(x) = -\cos(x)$ $\quad f'''(0) = -1$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \frac{0}{1} x^0 + \frac{1}{1!} x^1 + \frac{0}{2!} x^2 + \frac{-1}{3!} x^3 + \cdots$$

$$\boxed{\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}}$$

Maclaurin Series:

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

4 Finding and using Taylor Series

Ways to find Taylor Series that aren't computing all the derivatives:

- Composing with a known series. $\frac{1}{1-x}$ $\sin(x)$ $\cos(x)$ e^x

$$e^{x^4} = \sum_{n=0}^{\infty} \frac{(x^4)^n}{n!}$$

- Multiplying or dividing known series by powers of x

$$\frac{x^2}{1-x} = x^2 \sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} x^{n+2}$$

- Differentiating or integrating known series

$$\arctan(x) = \int \frac{1}{1+x^2}\,dx$$

- Multiplying two partial series

- Multiplying Power Series

$$e^x \sin(x) = \left(1 + x + \frac{x^2}{2} + \cdots\right)\left(x - \frac{x^3}{6} + \frac{x^5}{120} \cdots\right)$$

$$x + x^2 + \frac{x^3}{3} + \cdots$$

Example: Find the Maclaurin series for $f(x) = x \sin(x^2)$

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

$$\sin(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (x^2)^{2n+1}$$

$$\boxed{x \sin(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{4n+3}}$$

$$x \sin(x^2) = -\frac{1}{2}\left(\cos(x^2)\right)'$$

5 Other uses of Taylor Series

There are several other things we can do with Taylor Series:

- Approximating integrals

$$\int_0^1 e^{-x^2}\,dx$$

→ Numerical Integration

→ Taylor Series.

- Computing limits that would otherwise require L'Hopital.

• Would require many iterations to get what the limit converges to.

→ A lot easier with Taylor Series.

Example: Compute the limit

$$\lim_{x\to 0} \frac{1 + x + \frac{x^2}{2} - e^x}{2x^3} \qquad = \frac{0}{0}$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots$$

$$= \underline{1 + x + \frac{x^2}{2}} + \frac{x^3}{6} + x^4(\text{stuff})$$

$$1 + x + \frac{x^2}{2} - e^x = -\frac{x^3}{6} - x^4(\text{stuff})$$

$$\lim_{x\to 0} \frac{1 + x + \frac{x^2}{2} - e^x}{2x^3} = \lim_{x\to 0} \frac{-\frac{x^3}{6} - x^4(\text{stuff})}{2x^3}$$

$$= \lim_{x\to 0} \frac{-1/6 - x(\text{stuff})}{2}$$

$$\boxed{= -1/12}$$