Power Series

Learning Goals

- Identify a power series
- Find the interval and radius of convergence for a power series
- Add and multiply two power series together
- Find the power series representation of a function using a known power series
- Find the function represented by a given power series
- Differentiate and integrate a power series

Contents

1 Definition of Power Series

This section leads into power series, which is a very important tool in a lot of physical applications. A lot of well-known functions can be written as power series, and certain functions, like Bessel functions (which are very common in physics applications), can *only* be written as power series.

- Differential Equations
- Bessel Functions

Definition: A **power series** with *center* c is an infinite series of the form

$$F(x) = \sum_{n=0}^{\infty} a_n (x-c)^n.$$

a_n · sequence of coefficients.
c · center of the power series.

Think of as "infinite degree polynomial"

$c=0$ $\quad F(x) = \sum_{n=0}^{\infty} a_n x^n$

Operations on Power Series

Power Series can formally be treated like polynomials, but care is needed at each step.

Take two power series centered at zero

$$F(x) = \sum_{n=0}^{\infty} a_n x^n \qquad G(x) = \sum_{n=0}^{\infty} b_n x^n$$

$$F(x) + G(x) = \sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} b_n x^n$$

$$= \sum_{n=0}^{\infty} (a_n + b_n) x^n$$

$$F(x)G(x) = \left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

$$= \left(a_0 + a_1 x + a_2 x^2 + \dots\right)\left(b_0 + b_1 x + b_2 x^2 + \dots\right)$$

$$= a_0 b_0 \underline{x}^0 + (a_0 b_1 + a_1 b_0)\underline{x}^1 + (a_0 b_2 + a_1 b_1 + a_2 b_0)x^2$$

$$+ \dots$$

To get to x^k, add up products of all pairs of numbers that add to k.

$$\boxed{\left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_k b_{n-k}\right)x^n}$$

2 Convergence of Power Series

Power Series are a type of infinite series, so we need to talk about convergence of this series. However, now convergence will depend on the value of x.

Example: For what values of x does the series $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{3^n}$ converge?

· Do the test normally, but have x in answer.

Ratio Test

$$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{x^{n+1}/3^{n+1}}{x^n/3^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{x}{3} \right| = \left| \frac{x}{3} \right|$$

· $\rho < 1$ means absolute convergence $\rho = \frac{|x|}{3} < 1$

So absolute convergence if $|x| < 3$

· $\rho > 1$ is when $|x| > 3$ $\rightarrow$ Divergence.

$|x| = 3$? More direct Method.

$x=3$ $\quad \displaystyle\sum_{n=0}^{\infty} \frac{3^n}{3^n} = \sum_{n=0}^{\infty} 1$ diverges

$x=-3$ $\quad \displaystyle\sum_{n=0}^{\infty} \frac{(-3)^n}{3^n} = \sum_{n=0}^{\infty} (-1)^n$ diverges

The series $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{3^n}$ converges for

x in $\boxed{(-3, 3)}$

It turns out the type of answer we got for the previous example is not a coincidence. The fact that this was an interval is *always* what happens.

Definition: For any power series,

$$F(x) = \sum_{n=0}^{\infty} a_n (x-c)^n.$$

there is an interval of x values on which it converges. This is called the **interval of convergence** for that power series, and is an interval centered around c.

$c=0$ $(-3,3)$

Interval of convergence $(c-R, c+R)$

for some number R.

$R=0$, R could be infinite,

or R positive number.

R is the radius of convergence

Radius of Convergence

$$F(x) = \sum_{n=0}^{\infty} a_n (x-c)^n.$$

This series always converges at c.

- Possible for the series to only converge at c. $R=0$
- Possible for the series to converge for all x. $(-\infty, \infty)$ $R=\infty$
- Possible to converge on some interval and diverge elsewhere. R some number

$(c-R, c+R)$

→ Endpoints can be open or closed.

Types of Convergence

We know the series converges on this interval. Is more always true?

If $|x-c| < R$ then the series converges absolutely.

If $|x-c| > R$ the series diverges.

If $|x-c| = R$ anything can happen

Must check specifically

- → Absolute convergence
- → Divergence
- → Conditional convergence.

→ Find R by Ratio test

- Put x in the expression
- $R = 1/p$ ← Limit from Ratio test if you ignore x^n.

Example: Where does the power series $\sum_{n=1}^{\infty} \frac{x^n}{n}$ converge?

Ratio Test

$$\rho = \lim_{n\to\infty} \left| \frac{x^{n+1}/n+1}{x^n/n} \right| = \lim_{n\to\infty} |x| \cdot \frac{n}{n+1}$$

$$= |x|$$

Converges $|x| < 1$ Diverges for $|x| > 1$

Endpoints?

$x = 1$ $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges

$x = -1$ $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ Converges by Alternating Series test.

So: Converges on $\boxed{[-1, 1)}$.

3 Power Series Expansions $F(x) = \sum_{n=0}^{\infty} a_n x^n$

We have these power series. They may converge, but we might not know what they actually converge to (as functions). If we know what the function that we get as a result is, then we say that this power series is a **power series expansion** of that function.

Example: What do we know about the power series $\sum_{n=0}^{\infty} x^n$?

Ratio Test: $|x| < 1$

For any $|r| < 1$ $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$

$$\boxed{\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}}$$ $|x| < 1$

How can we find power series expansions?

- It's hard.

We can manipulate our one trick into something that works.

$$\frac{1}{1-u} = \sum_{n=0}^{\infty} u^n$$

$$\frac{x}{1-4x^2} = x \cdot \frac{1}{1-\underbrace{4x^2}}$$

$$= x \sum_{n=0}^{\infty} (4x^2)^n$$

$$= \boxed{\sum_{n=0}^{\infty} 4^n x^{2n+1}}$$

Example: Find a power series expansion for the function $\frac{1}{1+2x^3}$. Where is this expansion valid?

$$\frac{1}{1-u} = \sum_{n=0}^{\infty} u^n \qquad |u| < 1$$

$$\frac{1}{1+2x^3} = \frac{1}{1-(-2x^3)} = \sum_{n=0}^{\infty} (-2x^3)^n \qquad |-2x^3| < 1$$

$$= \boxed{\sum_{n=0}^{\infty} (-2)^n x^{3n}} \qquad |x^3| < 1/2$$

$$|x| < 1/\sqrt[3]{2}$$

4 Differentiation and Integration of Power Series

The main reason power series are so useful is because of the following properties:

Theorem. *Assume that the power series* $F(x) = \sum_{n=0}^{\infty} a_n(x-c)^n$ *has radius of convergence* $R > 0$ *(or* ∞*). Then*

F(x) is both differentiable and integrable on $(c-R, c+R)$

[or $(-\infty, \infty)$ if $R = \infty$]

AND we can differentiate and integrate term by term.

$\frac{d}{dx}(x-c)^n = n(x-c)^{n-1}$

$$F'(x) = \sum_{n=1}^{\infty} n a_n (x-c)^{n-1}$$

$$\int F(x)\,dx = \sum_{n=0}^{\infty} \frac{a_n}{n+1}(x-c)^{n+1} + C$$

Radius of convergence for these is also R.

Example: Find a power series expansion for $\arctan x$. Where is this expansion valid?

$$\frac{d}{dx}\left(\arctan(x)\right) = \frac{1}{1+x^2}$$

$$\frac{1}{1-(-x^2)} \quad \frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-x^2)^n \qquad |x^2| < 1$$

$$|x| < 1$$

$$\arctan(x) = \int \frac{1}{1+x^2}\,dx = \int \sum_{n=0}^{\infty} (-1)^n x^{2n}\,dx$$

$$= \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}} \qquad |x| < 1$$

5 A Famous Example

Let's consider a new power series:

$$F(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

What do we know about this power series?

Ratio Test

$$\rho = \lim_{n\to\infty} \left|\frac{x}{n}\right| = 0$$

$$R = \infty$$

$\rightarrow$ Power Series converges everywhere.

$$F'(x) = \frac{d}{dx}\left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right)$$

$$= \sum_{n=1}^{\infty} \frac{\cancel{n}\cdot x^{n-1}}{\cancel{n}!} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!}$$

$m = n-1$

$$= \sum_{m=0}^{\infty} \frac{x^m}{m!} = F(x)$$

$$\boxed{F(x) = e^x}$$

Example: Find a power series expansion for

$$f(x) = \frac{1}{x^2 - 4x + 4}. = \frac{1}{(x-2)^2}$$

$$\frac{d}{du}\frac{1}{1-u} = \frac{1}{(1-u)^2}$$

$$\frac{1}{2-x} = \frac{1}{2}\,\frac{1}{1-\frac{x}{2}} = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^n \qquad \left|\frac{x}{2}\right|<1$$

$$\frac{1}{(2-x)^2} = \frac{d}{dx}\left(\frac{1}{2-x}\right) = \frac{d}{dx}\left(\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n x^n\right)$$

$$= \frac{1}{2}\sum_{n=1}^{\infty} n\left(\frac{1}{2}\right)^n x^{n-1} \qquad |x|<2$$

$$= \sum_{m=0}^{\infty}(m+1)\left(\frac{1}{2}\right)^{m+2} x^m$$