Ratio and Root Tests

Learning Goals

- Determine if a series converges or diverges using the ratio test
- Determine if a series converges or diverges using the root test
- Choose an appropriate convergence test for a series
- Determine if a series converges or diverges using any method/test

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1 Ratio Test

This section covers two more tests for evaluating whether or not a series converges or diverges. They work for series with both positive and negative terms, but sort of ignore that fact by taking absolute values first. The extra benefit they have is that they do not require the series to alternate in order to give a result.

Theorem (Ratio Test). Let $\sum a_n$ be a series that we want to analyze. Assume that the following limit exists

$\rho = \lim_{n \to \infty} \left \frac{a_{n+1}}{a_n} \right $ Then If $q \leq 1$, this series	z s n=1 absolutely.
IF p71, this series	fiverges.
IF p=1, the test i	s inconclusive.

The idea as to why this works is direct comparison to a geometric series, which we will illustrate later.

Example: Does
$$\sum_{n=1}^{\infty} \frac{n^2}{32}$$
 converge?
Ratio Test
 $p = \left| \frac{n}{n} \right| \left| \frac{\Omega_{n+1}}{\Omega_n} \right| = \left| \frac{n}{n^{340}} \right| \frac{(n+1)^2/3^{n+1}}{n/3^n}$
 $= \left| \frac{n}{n^{340}} \right| \left| \frac{\Omega_{n+1}}{n^2} \cdot \frac{3^n}{3^{n+1}} \right|$
 $= \left| \frac{n}{n^{340}} \right| \left| \frac{(n+1)^2}{n^2} \cdot \frac{3^n}{3} \right|$
 $= \frac{1}{3} \left| \frac{n}{n^{340}} \right| \left| \frac{(n+1)^2}{n^2} \right| = \left| \frac{1}{3} \right|$
Since $p < l$, this series converges by the ratio test.

2 Root Test

The other test we have in this section is the Root Test. It does the same thing, but with roots instead of ratios.

Theorem (Root Test). Let $\sum a_n$ be a series that we want to analyze. Assume that the following limit exists

Then
Then

$$T \notin \{ \{ \{ \} \}, \{ \} \}, \{ \} \}$$
 an converges absolutely.
 $n = 1$
 $n = 1$
 $T \notin \{ \}, \{ \} \}, \{ \} \}$ an diverges.
 $n = 1$
 $T \notin \{ \} = 1, \{ \} \}$ is inconclusive.

The idea of the proof is the same, but it is more complicated.

Example: Does
$$\sum_{n=1}^{\infty} \left(\frac{2n}{n+4}\right)^n$$
 converge or diverge?

$$\frac{Root}{Root} \xrightarrow{Test}_{n \neq s} \left(\frac{2n}{n+4}\right)^n$$

$$t = \lim_{n \to \infty} \left(1 \ln 1\right) = \lim_{n \to \infty} \left(\frac{2n}{n+4}\right)^n$$

$$= \lim_{n \to \infty} \frac{2n}{n+4} = 2$$

$$= \lim_{n \to \infty} \frac{2n}{n+4} = 2$$
Since $f > 1$, the series $\sum_{n=1}^{\infty} \left(\frac{2n}{n+4}\right)^n \frac{1}{2^n} \frac{1}{2^n}$
by the cost test.

3 Proof of Ratio Test
Direct Comparison to
a Geometric Series.
A ssume
$$p < 1$$
. Pick $r = \frac{p+1}{2}$
 $p < r < 1$.
A ssume $p < 1$. Pick $r = \frac{p+1}{2}$
 $p < r < 1$.
There is N >> that if $n \ge N$ | $\frac{a_{n+1}}{a_n} | < r$
 $|a_{n+1}| \le r |a_n|$
 $|a_{n+2}| \le r |a_n|$
My Series Geometric that Converges
 $\sum_{n=1}^{N} |a_n|^2$
 $\sum_{n=1}^{N} |a_n|^2$
 $\sum_{n=1}^{N} |a_n|^2$

Example: Does
$$\sum_{n=2}^{\infty} \frac{5^n}{n!}$$
 converge or diverge?

Ratio Test
 $p = \lim_{n \to \infty} 0$ and $= \lim_{n \to \infty} \frac{5^{n+1}}{5^n/n!}$
 $p = \frac{1}{n \to \infty} 0$ $\frac{5^{n+1}}{5^n} \frac{n!}{(n!)!}$
 $= \lim_{n \to \infty} \frac{5^{n+1}}{5^n} \frac{n!}{(n!)!}$
 $= \lim_{n \to \infty} \frac{5^n}{n!} \frac{5^n}{(n!)!}$
 $= \lim_{n \to \infty} \frac{5^n}{n!} \frac{5^n}{(n!)!}$
Since p<1, the series $\sum_{n=2}^{\infty} \frac{5^n}{n!}$ converges
 $\sum_{n=2}^{\infty} \frac{5^n}{n!} \frac{5^n}{(n!)!}$

4 Choosing Tests

How do we choose which test to use in a given case? Which is the best order to attempt these tests to make the process as simple as possible?

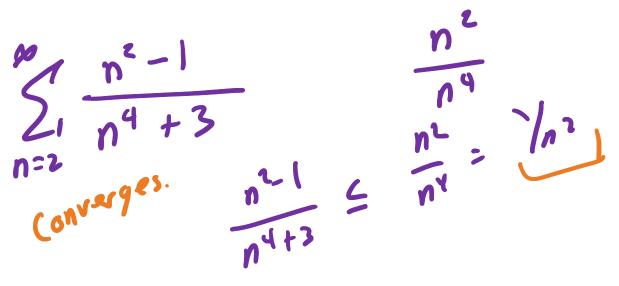
First, try the nth term divergence test. Remember this can **only** tell you that a series diverges, not that it converges.

If lim an #0, then diverges. Says that Jiverges Series K Only Limit is 1 so diverges. $SM(n) + n^2$ n: |

If a series does not have all positive terms, you have basically two options:

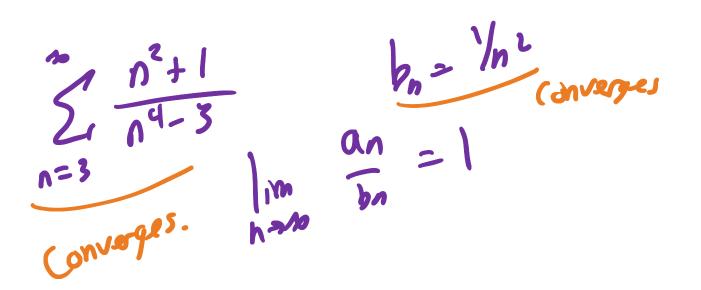
If a series has positive terms (or you made it that way by taking absolute values) now we have more options.

Is there a way I con make my Series bigger (smaller) and get (a) Direct Comparison Test Convergence (divergence)? - In general, Compare to g-series. -> Dropping terms from numerator or denomnator.

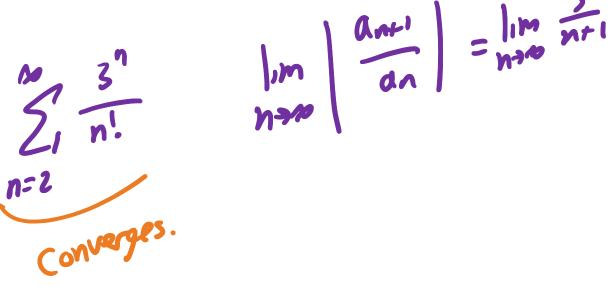


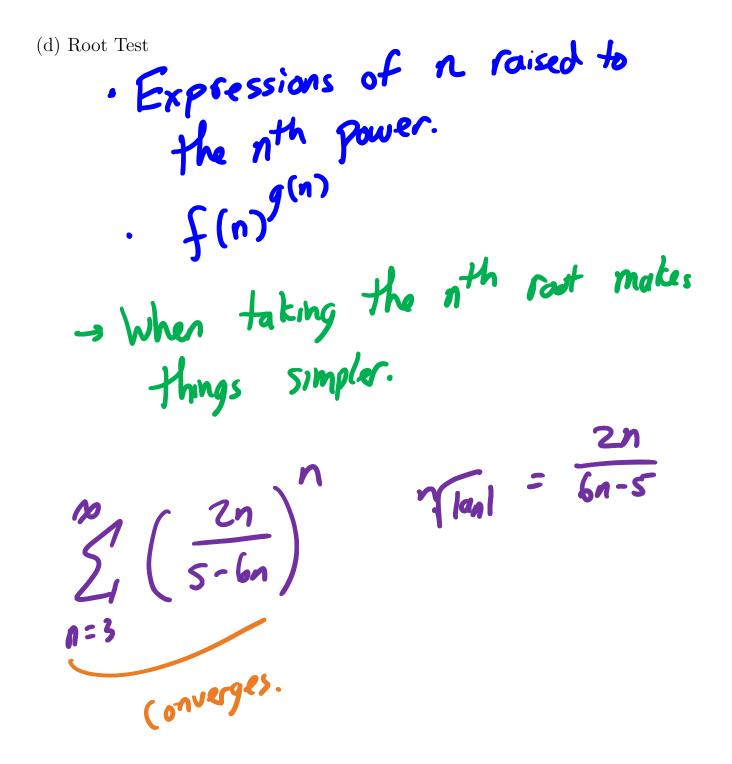
(b) Limit Comparison test

. Are there dominant terms in the numerator and denominator that will help me get convergence? - Don't have to worry about bigger or smaller. - Need to pick something that approximates my Series.



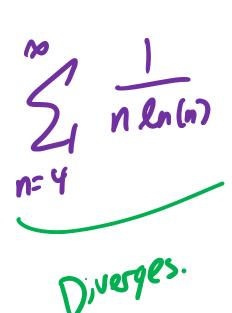
(c) Ratio Test





(e) Integral Test · If I Convert n to X, Con I

integrale this? - Works every time IF you confind the integral.



 $\frac{1}{\chi e_A(x)} dx$ $U = \ln f(x)$ 1 du diverges 66

Examples: Analyze each of the following series and determine whether they converge or diverge.

$$\sum_{n=1}^{\infty} \frac{1}{(n+2)^n}$$
Direct Componison: $n+2 > 2$

$$\lim_{n \to \infty} C 2^{-n}$$

$$\sum_{n=2}^{\infty} \frac{1}{n^2 - \sqrt{n}}$$
Limit Composison $b_n = \frac{1}{n^2}$

$$\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{\frac{1}{\sqrt{n^2 - \sqrt{n}}}}{\frac{1}{\sqrt{n^2}}}$$

$$= \lim_{n \to \infty} \frac{n^2}{n^2 - \sqrt{n}} = 1$$
Since $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n^2}}$ Converses so does
$$\sum_{n=2}^{\infty} \frac{1}{\sqrt{n^2 - \sqrt{n}}}$$

$$\sum_{n=1}^{\infty} \frac{n!}{(2n)!}$$

Ratio Test $P = \lim_{h \to \infty} \frac{\binom{n+1}{2(n+1)!}}{n!}$ $= \lim_{n \to \infty} \frac{(n+1)!}{n!} \frac{(2n)!}{(2n+2)!}$ $= \lim_{n \to \infty} \frac{n+1}{n+1} \left(\frac{1}{2n+2} \right)$ $= \lim_{h \to \infty} \frac{n+1}{(2n+1)(2n+2)} = 0 < 1$ <u>Sini</u> (on verges.

$$\sum_{n=1}^{\infty} \frac{3^n}{n!} \quad \text{Converges}.$$

$$\frac{Ratio}{p = 1} \frac{Jest}{m - n} \quad \frac{3^n / n!}{3^n / n!}$$

$$= \frac{1}{m + n} \quad \frac{3}{n + 1} \quad \frac{3}{n - 2} \quad \frac{3}{n + 1} \quad \frac{3}{n - 2}$$