Ratio and Root Tests

Learning Goals

- Determine if a series converges or diverges using the ratio test
- Determine if a series converges or diverges using the root test
- Choose an appropriate convergence test for a series
- Determine if a series converges or diverges using any method/test

Contents

1 Ratio Test

This section covers two more tests for evaluating whether or not a series converges or diverges. They work for series with both positive and negative terms, but sort of ignore that fact by taking absolute values first. The extra benefit they have is that they do not require the series to alternate in order to give a result.

Theorem (Ratio Test)**.** *Let $\sum a_n$ be a series that we want to analyze. Assume that the following limit exists*

$$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \geq 0$$

Then

If $\rho < 1$, this series $\sum_{n=1}^{\infty} a_n$ converges absolutely.

If $\rho > 1$, this series diverges.

If $\rho = 1$, the test is inconclusive.

The idea as to why this works is direct comparison to a geometric series, which we will illustrate later.

Example: Does $\sum_{n=1}^{\infty} \frac{n^2}{3^n}$ converge?

Ratio Test

$$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)^2/3^{n+1}}{n/3^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(n+1)^2}{n^2} \cdot \frac{\cancel{3^n}}{3^{\cancel{n}+1}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(n+1)^2}{n^2} \cdot \frac{1}{3} \right|$$

$$= \frac{1}{3} \underbrace{\lim_{n\to\infty} \left| \frac{(n+1)^2}{n^2} \right|}_{1} = \boxed{1/3}$$

Since $\rho < 1$, this series converges by the ratio test.

2 Root Test

The other test we have in this section is the Root Test. It does the same thing, but with roots instead of ratios.

Theorem (Root Test)**.** *Let* $\sum a_n$ *be a series that we want to analyze. Assume that the following limit exists*

$$t = \lim_{n\to\infty} \sqrt[n]{|a_n|}$$

Then

If $t < 1$, $\sum_{n=1}^{\infty} a_n$ converges absolutely.

If $t > 1$, $\sum_{n=1}^{\infty} a_n$ diverges.

If $t = 1$, test is inconclusive.

The idea of the proof is the same, but it is more complicated.

Example: Does $\displaystyle\sum_{n=1}^{\infty}\left(\frac{2n}{n+4}\right)^{n}$ converge or diverge?

Root Test

$$t = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \cancel{\sqrt[n]{}}\sqrt{\left(\frac{2n}{n+4}\right)^{\cancel{n}}}$$

$$= \lim_{n\to\infty} \frac{2n}{n+4} = \boxed{2}$$

Since $t > 1$, the series $\displaystyle\sum_{n=1}^{\infty}\left(\frac{2n}{n+4}\right)^{n}$ diverges by the root test.

3 Proof of Ratio Test

Direct Comparison to a geometric Series.

$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$

"almost geometric at the end"

$\rho > r > 1$

Assume $\rho < 1$. Pick $r = \frac{\rho+1}{2}$ $\quad \rho < r < 1$.

There is N so that if $n \geq N$ $\left| \frac{a_{n+1}}{a_n} \right| < r$

$|a_{N+1}| \leq r|a_N|$

$|a_{N+2}| \leq r|a_{N+1}| < r^2 |a_N|$

For any k $\underbrace{|a_{N+k}|}_{\text{My Series}} < \underbrace{r^k |a_N|}_{\text{Geometric}}$ that converges

$\sum_{k=1}^{\infty} |a_{N+k}|$ converges by direct comparison with $r^k|a_N|$

So $\sum_{n=1}^{\infty} a_n$ converges absolutely.

Example: Does $\displaystyle\sum_{n=2}^{\infty} \frac{5^n}{n!}$ converge or diverge?

Ratio Test

$$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{5^{n+1}/(n+1)!}{5^n/n!} \right|$$

$$= \lim_{n\to\infty} \left| \frac{5^{n+1}}{5^n} \cdot \frac{n!}{(n+1)!} \right|$$

$$= \lim_{n\to\infty} \left| \frac{5}{n+1} \right| = \boxed{0}$$

Since $\rho < 1$, the series $\displaystyle\sum_{n=2}^{\infty} \frac{5^n}{n!}$ converges absolutely by the ratio test.

4 Choosing Tests

How do we choose which test to use in a given case? Which is the best order to attempt these tests to make the process as simple as possible?

First, try the nth term divergence test. Remember this can **only** tell you that a series diverges, not that it converges.

If $\lim_{n \to \infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges.

* Only says that a series diverges *

$$\sum_{n=1}^{\infty} \frac{\sin(n) + n^2}{n^2 + 1}$$

Limit is 1 so diverges.

If a series does not have all positive terms, you have basically two options:

1. Alternating Series Test
   - → Requires the series to be alternating.
   - → Pretty easy to apply after that.

$\sum_1 (-1)^n b_n$ $b_n \geq 0$, decreasing
$b_n \to 0$

$\sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ converges by Alternating Series Test.

2. Take the absolute value of the terms and apply "positive terms" tricks.
   - → Looking for absolute convergence.

If a series has positive terms (or you made it that way by taking absolute values) now we have more options.

(a) Direct Comparison Test

Is there a way I can make my series bigger (smaller) and get convergence (divergence)?

→ In general, compare to p-series.

→ Dropping terms from numerator or denominator.

$$\sum_{n=2}^{\infty} \frac{n^2-1}{n^4+3} \qquad \frac{n^2}{n^4}$$

Converges.

$$\frac{n^2-1}{n^4+3} \le \frac{n^2}{n^4} = \underbrace{1/n^2}$$

(b) Limit Comparison test

- Are there dominant terms in the numerator and denominator that will help me get convergence?
  - → Don't have to worry about bigger or smaller.
  - → Need to pick something that approximates my series.

$$\underbrace{\sum_{n=3}^{\infty} \frac{n^2+1}{n^4-3}}_{\text{Converges.}} \qquad \underbrace{b_n = 1/n^2}_{\text{converges}}$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 1$$

(c) Ratio Test

- Factorials
- Polynomials and numbers raised to the n^{th} power.

→ Make the expression nice to take the limit of.

$$\sum_{n=2}^{\infty} \frac{3^n}{n!}$$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{3}{n+1}$$

Converges.

(d) Root Test

- Expressions of n raised to the n^{th} power.
- $f(n)^{g(n)}$

→ When taking the n^{th} root makes things simpler.

$$\sum_{n=3}^{\infty} \left(\frac{2n}{5-6n} \right)^n \qquad \sqrt[n]{|a_n|} = \frac{2n}{6n-5}$$

converges.

(e) Integral Test

- If I convert n to x, can I integrate this?

→ Works every time IF you can find the integral.

$$\sum_{n=4}^{\infty} \frac{1}{n\ln(n)}$$

Diverges.

$$\int_2^{\infty} \frac{1}{x\ln(x)}\,dx$$

$u = \ln(x)$

$$\int_{\ln 2}^{\infty} \frac{1}{u}\,du$$ diverges

Examples: Analyze each of the following series and determine whether they converge or diverge.

$$\sum_{n=1}^{\infty} \frac{1}{(n+2)^n}$$

$n+2 > 2$

Direct Comparison:

$$\frac{1}{(n+2)^n} < \underbrace{2^{-n}}_{\text{Geometric that Converges}}$$

So this converges.

Root Test

$$t = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{1}{n+2} = 0$$

$t < 1$ so Converges.

$$\sum_{n=2}^{\infty} \frac{1}{n^2 - \sqrt{n}}$$

$b_n = 1/n^2$

Limit Comparison

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1/(n^2-\sqrt{n})}{1/n^2}$$

$$= \lim_{n\to\infty} \frac{n^2}{n^2-\sqrt{n}} = 1$$

Since $\sum_{n=2}^{\infty} 1/n^2$ converges so does $\sum_{n=2}^{\infty} \frac{1}{n^2-\sqrt{n}}$

$$\sum_{n=1}^{\infty} \frac{n!}{(2n)!}$$

Ratio Test

$$\rho = \lim_{n\to\infty} \left| \frac{(n+1)!/(2(n+1))!}{n!/(2n)!} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(n+1)!}{n!} \cdot \frac{(2n)!}{(2n+2)!} \right|$$

$$= \lim_{n\to\infty} \left| n+1 \cdot \frac{1}{(2n+1)(2n+2)} \right|$$

$$= \lim_{n\to\infty} \left| \frac{n+1}{(2n+1)(2n+2)} \right| = 0 < 1$$

So $\sum_{n=1}^{\infty} \frac{n!}{(2n)!}$ converges.

$$\sum_{n=1}^{\infty} \frac{3^n}{n!}$$ Converges.

Ratio Test

$$\rho = \lim_{n\to\infty} \left| \frac{3^{n+1} / (n+1)!}{3^n / n!} \right|$$

$$= \lim_{n\to\infty} \left| \frac{3}{n+1} \right| = 0$$