Conditional Convegence and Alternating Series

Learning Goals

- Understand the difference between absolute and conditional convergence
- Determine whether a series converges absolutely or conditionally
- Identify a series as an alternating series
- Use the Alternating Series Test to determine if a series converges
- Determine how many terms are needed to accurately approximate the sum of an alternating series

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1 Absolute and Conditional Convergence

Now, we want to start looking an series that don't necessarily have positive terms. Being able to handle series like this is really important for dealing with Taylor Series. We need a few definitions in order to handle these series.

Definition. We say that the series $\sum a_n$ **converges absolutely** if

$\sum |a_n|$ converges.

→ This is a series with positive terms.

Example: Does $\sum_{n=2}^{\infty} \frac{(-1)^n}{n^4}$ converge absolutely? What about $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$?

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{n^4} \rightarrow \sum_{n=2}^{\infty} \left| \frac{(-1)^n}{n^4} \right| = \sum_{n=2}^{\infty} \frac{1}{n^4}$$

→ Converges Absolutely

Converges by p-series.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \rightarrow \sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n} \right| = \sum_{n=1}^{\infty} 1/n$$

Diverges by p-series

→ Does not converge absolutely.

Theorem. *If $\sum |a_n|$ converges, then $\sum a_n$ also converges.*

or If $\sum_1 a_n$ converges absolutely, then $\sum_1 a_n$ converges.

Why does this make sense?

$\sum_1 |a_n|$

Going to 0 fast enough to converge

$\sum_1 a_n$

Making some +, some − can't make it diverge.

$$-|a_n| \leq a_n \leq |a_n|$$

$$0 \leq \underbrace{a_n + |a_n|}_{\text{Converges by Direct Comparison}} \leq 2|a_n|$$

This series will converge

But, this is not the only way to get convergence for a series that has terms that could be positive or negative. This gives rise to another definition.

Definition. An infinite series $\sum a_n$ **converges conditionally** if

$\sum_1 a_n$ converges but $\sum_1 |a_n|$ diverges.

☆ A series can not converge conditionally and converge absolutely. ☆

This part is hard, and the rest of this section covers a way to know if a series converges conditionally.

- We will develop one specific method for this.

2 Alternating Series

For series with not necessarily positive terms, there aren't too many ways to determine if the series converges conditionally. There are a few methods that we will not discuss in this class, but the main one that we will discuss here involves alternating series.

Definition. A series is alternating if it is of the form

$$\sum_{n=1}^{\infty} (-1)^n b_n$$

b_n is a positive sequence

$$\sum_{n=1}^{\infty} (-1)^{n+1} b_n$$

b_n positive

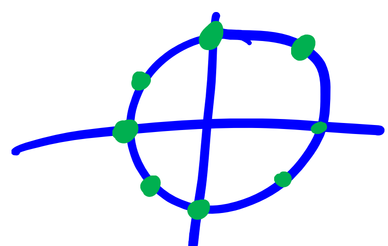
→ Terms need to alternate sign

Examples:

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{n^2} \qquad b_n = \frac{1}{n^2}$$

$$= \frac{1}{4} - \frac{1}{9} + \frac{1}{16} - \frac{1}{25} + \frac{1}{36} \cdots$$

$$\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n} = -1 + \frac{1}{2} - \frac{1}{3} \cdots$$



$$\sum_{n=1}^{\infty} \frac{\sin(n\pi/4)}{n^2}$$ $\times$ Not alternating

Theorem. ***Alternating Series Test*** *Assume that* $\{b_n\}$ *is a positive sequence that is decreasing and converges to zero:*

- $b_n \geq 0$,
- $b_{n+1} \leq b_n$ and
- $\lim_{n\to\infty} b_n = 0$

Then, the following alternating series converges:

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n \quad \text{converges.}$$

Furthermore, if S *is this sum, then*

$$0 < S < b_1$$

$$S_p < S < S_q$$

p even

Partial sums.

q odd

Example: Consider the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$. Does this converge? What can we say about the limit?

- This series is alternating $b_n = 1/n$

- $b_n \geq 0$, b_n decreasing

$$\lim_{n \to \infty} b_n = 0$$

Therefore, by the Alternating Series Test,

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \text{ converges.}$$

- Since $\sum_{n=1}^{\infty} 1/n$ diverges, $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges conditionally.

Know: $-1 < S < 0$

$-1 < S < -1/2$

3 Proof of Alternating Series Test

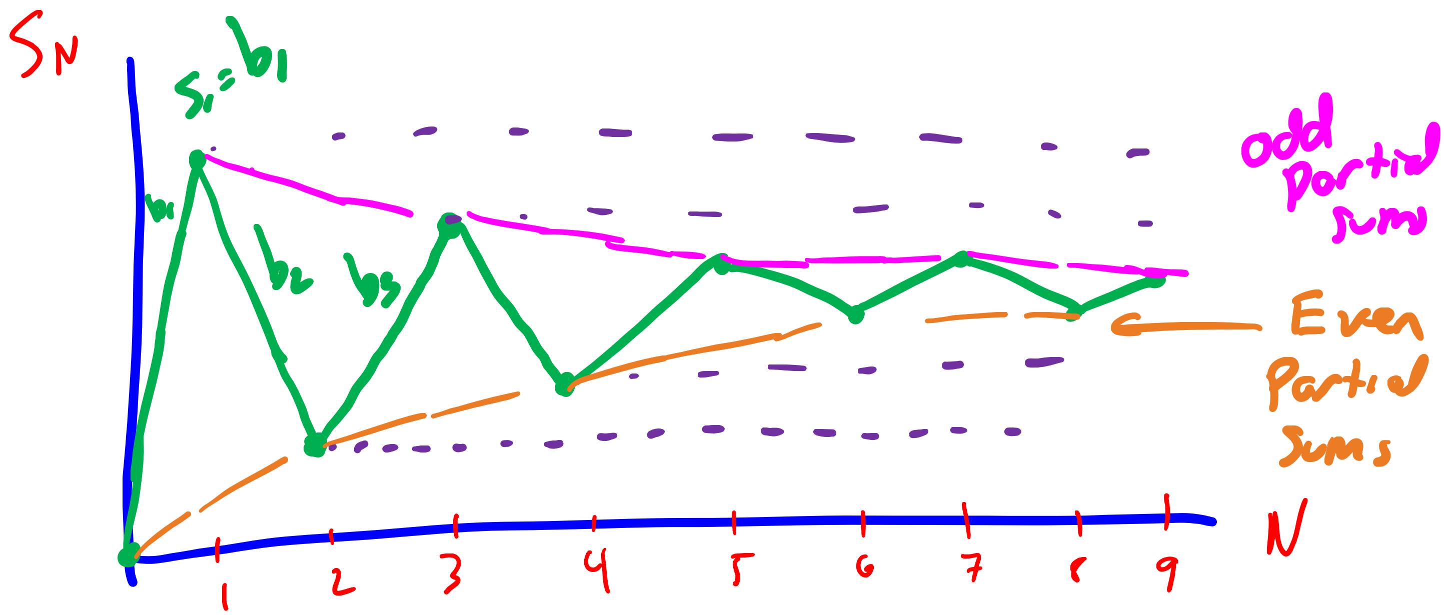
What is the idea of the proof?

$\sum_1 (-1)^{n-1} b_n$ Converges

$b_n \geq 0$

b_n decreasing

$b_n \to 0$



Convergence because odd partial sums are decreasing and bounded, and even partial sums are increasing and bounded.

$\Rightarrow$ Sequence of partial sums converges.

What is the point of this?

- Very easy way to determine if an alternating series converges.
  - Verify alternating
  - b_n positive, decreasing, $\to 0$.

- 'Alternating' isn't that rare
  - $\to$ Shows up in Power Series.

Example: Show that $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{2/3}}$ converges conditionally. Furthermore if S is the sum, then $-1 \le S \le 0$.

Convergence: Alternating series test

$$b_n = 1/n^{2/3}$$

- Positive
- Decreasing
- Goes to 0.

Therefore converges by Alternating Series Test.

Absolute Convergence: $\displaystyle\sum_{n=1}^{\infty} 1/n^{2/3}$ · Diverges by p-series.

So $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{2/3}}$ converges conditionally.

First term $= -1$ so $\boxed{-1 \le S \le 0}$

4 Error Bound on Alternating Series

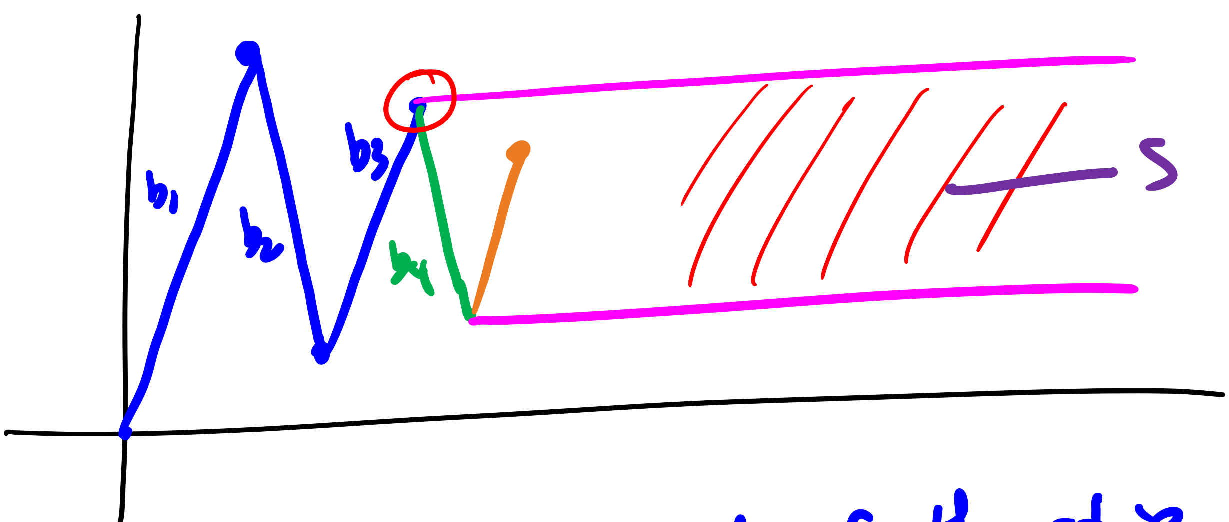
While the Alternating Series Test doesn't give us a way to compute the sums of series, it says they converge, but can get us pretty close.

Corollary. *Let* $S = \sum_{n=1}^{\infty}(-1)^{n-1}b_n$ *where* $\{b_n\}$ *is a positive decreasing sequence that converges to* 0. *Then* the series converges by AST.

And if $S_N = \sum_{n=1}^{N} (-1)^{n-1} b_n$, then

$$\boxed{|S - S_N| \leq b_{N+1}}$$

Idea:



→ Can never get outside of this strip.

Example: Analyze $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$. Does it converge? What else do we know about it? How close is S_{10} to the limit S?

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n \qquad b_n = \frac{1}{n}$$

- positive
- decreasing
- $\to 0$

So this series converges by Alternating Series Test. Furthermore, since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, we know this converges conditionally.

$$S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \cdots$$

$$0 \leq S \leq 1 \qquad \tfrac{1}{2} \leq S \leq 1$$

$$\boxed{|S - S_{10}| \leq b_{11} = \frac{1}{11}}$$

Example: Determine convergence or divergence of the following series:

$$\sum_{n=1}^{\infty} \frac{(-1)^n n}{\sqrt{n^2+1}} \qquad \sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n \ln n}$$

- Both are alternating series.

1) $b_n = \frac{n}{\sqrt{n^2+1}}$ $\lim_{n\to\infty} b_n = 1 \neq 0$

→ Alternating series test does not apply.

$\lim_{n\to\infty} (-1)^n \frac{n}{\sqrt{n^2+1}}$ does not exist

So this diverges by the n^{th} term divergence test.

2) $b_n = \frac{1}{n \ln n}$ b_n - positive - decreasing - $\to 0$

So $\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n \ln n}$ converges by Alternating Series Test.

Integral Test to $b_n = \frac{1}{n \ln n}$ → Diverges

Converges Conditionally