Series with Positive Terms

Learning Goals

- Identify a series as one with positive terms
- Determine convergence or divergence of a p-series
- Use the integral test to determine convergence or divergence of a series
- Use the Direct Comparison Test to determine convergence or divergence of a series
- Use the Limit Comparison Test to determine convergence or divergence of a series

Contents

1 Series with Positive Terms

In the last section, we talked about series and how to evaluate them. The only two tricks we really have for this is telescoping series or geometric series. However, not all series can be evaluated directly, but we still want to know if they converge or diverge. This section starts our discussion of 'Convergence Tests', determining whether or not a series converges without needing to compute the actual value.

Series with Positive Terms

Why start here?

1. It is easier

→ Every term being positive helps.

2. Can convert any series to one with positive terms using absolute values.

$$\sum_{n=1}^{\infty} a_n \rightsquigarrow \sum_{n=1}^{\infty} |a_n|$$

$$S_{N+1} - S_N = a_N > 0$$

What do we know if $a_n > 0$?

$$\underline{S_{N+1} > S_N}$$

$\hookrightarrow$ $\underline{\{S_N\} \text{ is increasing}}$

Monotone

Theorem 1

If $S_N = \sum_{n=1}^{N} a_n$ with $a_n > 0$ for all n

then if $\{S_N\}$ is <u>bounded</u>, it is <u>convergent</u>.

· If $\{S_N\}$ is <u>unbounded</u> then the series is <u>divergent</u>.

2 Integral Test and p-series

Now, we want to develop some convergence tests using the result from the last video.

Theorem 2: *Integral Test*

Let $a_n = f(n)$ where f is positive, continuous, and decreasing on $[1, \infty)$. Then

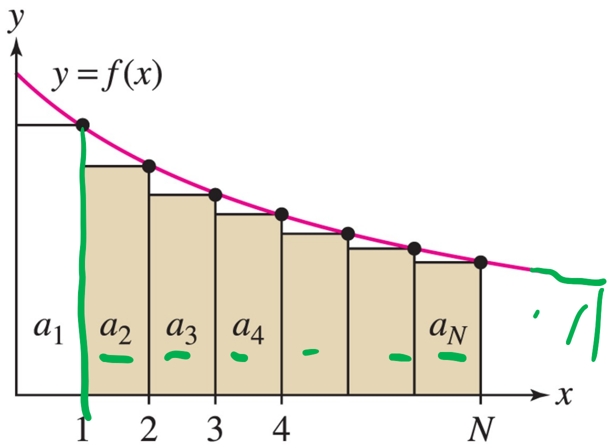
- If $\int_1^\infty f(x)\,dx$ converges, then $\sum_{n=1}^\infty a_n$ converges.
- If $\int_1^\infty f(x)\,dx$ diverges, then $\sum_{n=1}^\infty a_n$ diverges.

Proof:

$\int_1^N f(x)\,dx \le S_N$

$\downarrow \qquad\qquad \to \infty$

∞



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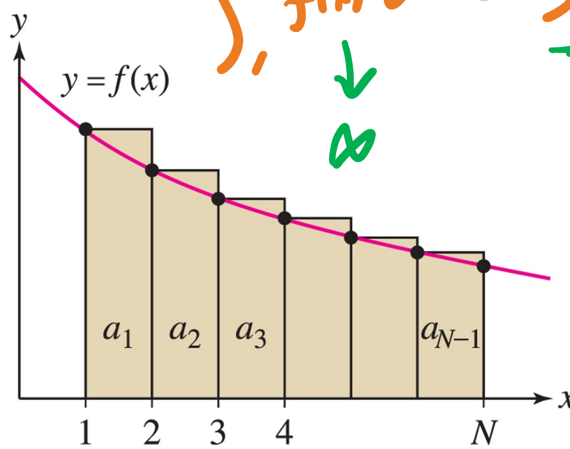

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$\int_1^\infty f(x)\,dx$ is a bound for the sequence of partial sums.

$$\sum_{n=2}^{N} a_n \le \int_1^\infty f(x)\,dx \quad \text{for all } N.$$

$$S_N \le \int_1^\infty f(x)\,dx + a_1 \;\to\; \text{Bounded sequence}$$

So S_N converges

If $\int_1^\infty f(x)\,dx$ does

Convergence of p-series

p-Series $\sum_{n=1}^{\infty} \frac{1}{n^p}$

Converge or diverge?

By the integral test, does $\int_1^{\infty} \frac{1}{x^p}\,dx$ converge or diverge?

- Converge $p > 1$
- Diverge $p \leq 1$

Same is true for the series

Example: Show that $\sum_{n=2}^{\infty} \frac{1}{n^2}$ converges but $\sum_{n=2}^{\infty} \frac{1}{n}$ diverges.

p-Series or Integral Test.

$\int_1^{\infty} \frac{1}{x^2}\,dx$ converges $= \boxed{1}$

but $\int_1^{\infty} \frac{1}{x}\,dx$ diverges.

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}}$$

3 Direct Comparison Test

Theorem 4: *Direct Comparison Test*

Let $\{a_n\}$ and $\{b_n\}$ be two sequences and assume there is some M so that $0 \leq a_n \leq b_n$ for all $n \geq M$

Then

- If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.
- If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.

Example: Does $\sum_{n=1}^{\infty} \frac{1}{n^3+1}$ converge?

Key trick: What do I compare to?

$$\frac{1}{n^3+1} \leq \frac{1}{n^3} \quad \text{for all } n$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges (p-series)

and $\frac{1}{n^3+1} \leq \frac{1}{n^3}$ for all n,

we know that by the direct comparison test

$\sum_{n=1}^{\infty} \frac{1}{n^3+1}$ converges

4 Limit Comparison Test

Theorem 5: *Limit Comparison Test*

- Comparing to a "similar" series.

Let $\{a_n\}$, $\{b_n\}$ be two positive sequences and assume $L = \lim_{n\to\infty} \frac{a_n}{b_n}$ exists

- If $L>0$ (and not infinite) then $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=1}^{\infty} b_n$ converges.

  → Always have the same behavior.

- If $L=\infty$ and $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} b_n$ converges.

- If $L=0$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

Example: Determine if $\sum_{n=1}^{\infty} \frac{n^2+2}{n^3+2n^2-3n+4}$ converges.

· Don't need an inequality

Use Limit Comparison

$$a_n = \frac{n^2+2}{n^3+2n^2-3n+4} \qquad b_n = \frac{n^2}{n^3} = \frac{1}{n}$$

$$L = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{n^2+2}{n^3+2n^2-3n+4}}{1/n}$$

$$= \lim_{n\to\infty} \frac{n^3+2n}{n^3+2n^2-3n+4} = 1 > 0$$

By LCT $\sum_{n=1}^{\infty} a_n$ behaves the same as $\sum_{n=1}^{\infty} b_n$

But $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} 1/n$ diverges

So $\sum_{n=1}^{\infty} \frac{n^2+2}{n^3+2n^2-3n+4}$ diverges