Infinite Series

Learning Goals

- Determine whether a series converges or diverges using the sequence of partial sums
- Evaluate a convergent series using algebraic properties
- Determine if a geometric series converges and if so find its sum
- Express repeating decimals as fractions using geometric series
- Evaluate a telescoping series

Contents

1 Infinite Series

Sometimes we can't write down exact decimal expansions for numbers, generally because they are irrational and so there's no finite representation for them. This is things like e or π or $\sin 1$. However, all of these examples here can be represented as **infinite series**.

$$e = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \cdots$$

$$= \sum_{n=0}^{\infty} 1/n!$$

$n! = n(n-1)(n-2)\cdots(2)(1)$

$$\sin(1) = 1 - \frac{1}{6} + \frac{1}{120} - \frac{1}{5040} + \cdots$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)!}$$

What does this equation mean?

$$e = \sum_{n=0}^{\infty} 1/n!$$

$$e = \lim_{N\to\infty} \sum_{n=0}^{N} 1/n!$$

- Series is the limit of the sequence of partial sums.

$a_n = 1/n!$

$a_n = 2^{-n}$

$$S_N = \sum_{n=0}^{N} a_n$$

Partial sums

Then $$\boxed{S = \sum_{n=0}^{\infty} a_n = \lim_{N\to\infty} S_N}$$

Definition: *Convergence of an Infinite Series*

We say that a series $\sum_{n=0}^{\infty} a_n$ converges if the limit of the partial sums exists, i.e. $\lim_{N\to\infty} \sum_{n=0}^{N} a_n$ exists.

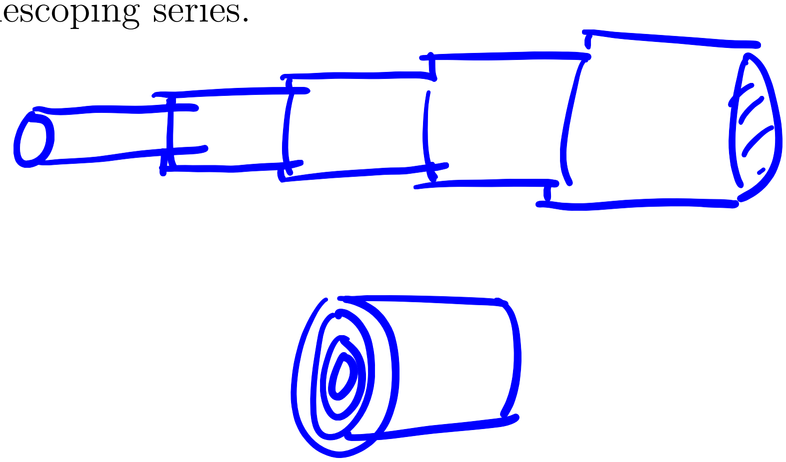
and we write

$$S = \lim_{N\to\infty} S_N = \sum_{n=0}^{\infty} a_n \qquad n=5$$

- If this limit does not exist, we say the series diverges.
- If $S_N \to \infty$, we can say the series diverges to ∞.

2 Telescoping Series

There are only a few series whose values we can actually compute. One of those is telescoping series.



Issue: How do you write S_N?

Telescoping series "collapse" making this part doable.

$$\sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1} = \left(\frac{1}{1} - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \frac{1}{4}\right)$$

$$\sum_{n=1}^{3} \frac{1}{n} - \frac{1}{n+1} = \frac{1}{1} - \frac{1}{4}$$

$$\sum_{n=1}^{N} \frac{1}{n} - \frac{1}{n+1} = S_N = \frac{1}{1} - \frac{1}{N+1}$$

$$\frac{1}{n}-\frac{1}{n+1}$$

Example: Investigate $\displaystyle\sum_{n=1}^{\infty} \frac{2}{n(n+2)}$.

$$\frac{2}{n(n+2)} = \frac{A}{n} + \frac{B}{n+2} = \frac{1}{n} - \frac{1}{n+2}$$

$$\sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+2}$$

$$S_N = \sum_{n=1}^{N} \frac{1}{n} - \frac{1}{n+2} = \left(1 - \cancel{\frac{1}{3}}\right) + \left(\frac{1}{2} - \cancel{\frac{1}{4}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{5}}\right) + \left(\cancel{\frac{1}{4}} - \cancel{\frac{1}{6}}\right) + \cdots + \left(\cancel{\frac{1}{N-1}} - \frac{1}{N+1}\right) + \left(\cancel{\frac{1}{N}} - \frac{1}{N+2}\right)$$

$$S_N = 1 + \frac{1}{2} - \underbrace{\frac{1}{N+1}}_{\to 0} - \underbrace{\frac{1}{N+2}}_{\to 0}$$

$$\lim_{N\to\infty} S_N = 3/2$$

$\boxed{\text{• Series Converges } \displaystyle\sum_{n=1}^{\infty} \frac{2}{n(n+2)} = 3/2}$

3 Geometric Series

The other main type of series where we can actually compute the value of is Geometric Series.

Geometric Sequence: $a_n = cr^n$

Geometric Series: $\sum_{n=0}^{\infty} cr^n$

$$S_N = c + cr + cr^2 + cr^3 + \dots + cr^N$$

$$rS_N = cr + cr^2 + cr^3 + cr^4 + \dots + cr^{N+1}$$

$$(1-r)S_N = c - cr^{N+1}$$

$$\boxed{S_N = c\,\frac{1-r^{N+1}}{1-r}}$$

Example: Investigate $\sum c \cdot r^n$ for $|r| < 1$.

$$S_N = c\left(\frac{1 - r^{N+1}}{1 - r}\right)$$

$$\lim_{N \to \infty} S_N \ ?$$

If $|r| \geq 1$, the r^{N+1} blows up as $N \to \infty$

$\to \lim_{N \to \infty} S_N$ Does not exist $\to$ Diverge

If $|r| < 1$ $\quad r^{N+1} \to 0$

$$\lim_{N \to \infty} S_N = c\left(\frac{1}{1-r}\right)$$ exists. Converges

$$\boxed{\sum_{n=0}^{\infty} cr^n = c \cdot \frac{1}{1-r}}$$

1 a

Example: Express 1.353535... as a fraction using a Geometric Series.

$$.\underbrace{35}\underbrace{35}\underbrace{35}\ldots$$

$35\left(\frac{1}{100}\right)^3$

$35 \cdot \frac{1}{100}$ $\qquad 35 \cdot \frac{1}{100} \cdot \frac{1}{100}$

$m = n-1$

$$.353535\ldots = \sum_{n=1}^{\infty} 35\left(\frac{1}{100}\right)^n$$

$$= \sum_{m=0}^{\infty} 35\left(\frac{1}{100}\right)^{m+1}$$

$$= \sum_{m=0}^{\infty} (.35)\left(\frac{1}{100}\right)^m = .35\left(\frac{1}{1-1/100}\right)$$

$$= .35\left(\frac{100}{99}\right)$$

$$= \frac{35}{99}$$

$$1.3\overline{535}\ldots = \boxed{\frac{134}{99}}$$

4 Limit Laws for Series

Theorem: Limit Laws for Series.

Assume that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge then $\sum_{n=1}^{\infty} a_n - b_n$, $\sum_{n=1}^{\infty} a_n + b_n$, $\sum_{n=1}^{\infty} ca_n$ all converge with

- $\sum_{n=1}^{\infty} a_n \pm b_n = \sum_{n=1}^{\infty} a_n \pm \sum_{n=1}^{\infty} b_n$
- $\sum_{n=1}^{\infty} ca_n = c \sum_{n=1}^{\infty} a_n$

$\underbrace{\sum_{n=1}^{\infty} \frac{2}{n(n+2)}}_{\text{Converges}}$ $\qquad \underbrace{\sum_{n=1}^{\infty} \frac{1}{n}}_{\text{Diverges}} - \underbrace{\sum_{n=1}^{\infty} \frac{1}{n+2}}_{\text{Diverges}}$

Example: Find the limit of the following series:

$$\sum_{n=0}^{\infty} 2\left(3^{-n} - 5^{-n}\right)$$

$$= \underbrace{2\sum_{n=0}^{\infty} 3^{-n}}_{\text{Geometric } r = 1/3} - \underbrace{2\sum_{n=0}^{\infty} 5^{-n}}_{\text{Geometric } r = 1/5}$$

$$2\left(\frac{1}{1-1/3}\right) - 2\left(\frac{1}{1-1/5}\right)$$

$$2\left(3/2\right) - 2\left(5/4\right) = \boxed{1/2}$$

5 Divergence Test

This provides our first example of deciding whether or not a series converges or diverges without needing to compute its value.

- Need to know if things converge
- Too complicated to evaluate by hand.
- Approximate with numerics.

Theorem: nth term Divergence Test

If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges

"If the terms don't go to zero, the series must diverge"

$\sum_{n=2}^{\infty} \frac{n}{4+n}$ diverges since $\lim_{n\to\infty} \frac{n}{4+n} = 1 \neq 0$

· Can NEVER tell you that a series converges, only diverges.

Non-Example: Investigate $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

$$a_n = \frac{1}{\sqrt{n}} \qquad \lim_{n\to\infty} a_n = 0$$

→ Divergence test tells me nothing!

$$S_N = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \cdots + \frac{1}{\sqrt{N}}$$

$$\geq \underbrace{\frac{1}{\sqrt{N}} + \frac{1}{\sqrt{N}} + \frac{1}{\sqrt{N}} + \frac{1}{\sqrt{N}} + \cdots + \frac{1}{\sqrt{N}}}_{N}$$

$$= \frac{N}{\sqrt{N}} = \sqrt{N}$$

$$S_N \geq \sqrt{N}$$

$\underline{S_N \to \infty}$ as $N \to \infty$

Therefore the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ $\boxed{\underline{\text{diverges}}}$