Sequences

Learning Goals

- Find the explicit formula for the nth term of an infinite sequence
- Determine if a sequence converges and if so find its limit
- Determine if a continuous function defined on a convergent sequence converges and if so finds its limit
- Use the Squeeze Theorem to find the limit of a sequence
- Determine if a sequence is bounded
- Use the properties of sequences to find limits of sequence given limits of other related sequences

Contents

1 Definition of a Sequence

This introduces the next unit of Calculus 2, which is that of sequences and series. This provides some of the foundation for what we have been doing previously and how we can apply these ideas to other areas.

- Taylor Series

→ Need some foundation to get there.

Definition: A sequence $\{a_n\}$ is

an ordered list of numbers, defined as a function on a set of increasing integers.

→ a_n gives the actual sequence "terms" of the sequence

→ n is the index

. where you are in the sequence

a_1 a_{10} $a_{13} - a_5$

→ Don't have to start at 1.

Examples:

$a_n = 2^{-n} \quad n \geq 0$

$\{1, 1/2, 1/4, 1/8, 1/16, \dots\}$

"Explicit Formula"

$a_5 = 1/32$

Fibonacci Numbers $1, 1, 2, 3, 5, 8, 13 \dots$

$a_1 = 1 \quad a_2 = 1 \quad a_n = a_{n-1} + a_{n-2}$

$n \geq 3$

"Recursively defined sequence"

$$b_n = \frac{f(1/n) - f(0)}{1/n}$$

- Approximations to $f'(0)$.

Definition: We say that $\{a_n\}$ **converges to a limit** L and write $\lim_{n\to\infty} a_n = L$ if "as $n \to \infty$, a_n gets closer to L"

For any $\varepsilon > 0$, there is an N so that for all $n \geq N$ $|a_n - L| < \varepsilon$.

→ "diverges" if not.

→ "diverges to ∞"

Theorem: If $a_n = f(n)$ and $\lim_{x\to\infty} f(x)$ exists, then $\lim_{n\to\infty} a_n = \lim_{x\to\infty} f(x)$

→ Highest power for rational functions

→ Behavior of exponentials.

Example: Let $a_n = 1 + \frac{1}{n}$. What is $\lim_{n\to\infty} a_n$?

$$\lim_{n\to\infty} a_n = \lim_{x\to\infty} f(x) \qquad f(x) = 1 + \frac{1}{x}$$

$$\boxed{\lim_{n\to\infty} a_n = 1}$$

2 Geometric and Recursive Sequences

Geometric Sequences

$$a_n = c\,r^n \quad \text{for } n \geq 0$$

$a_n = 2 \cdot 3^{-n}$ $n \geq 0$ $\qquad b_n = \frac{1}{2} 5^n$ $n \geq 0$

$a_1 = \frac{2}{3}$ $\quad a_2 = \frac{2}{9}$ $\quad a_3 = \frac{2}{27}$, ...

Do geometric sequences have limits?

$$\lim_{n \to \infty} c\,r^n = \lim_{x \to \infty} c\,r^x$$

If $0 < r < 1$ $\qquad c\,r^x \to 0$ as $x \to \infty$

$$\lim_{n \to \infty} c\,r^n = 0$$

If $r = 1$ $\qquad \lim_{n \to \infty} c(1)^n = c$

If $r > 1$ $\qquad c\,r^x \to \infty$, sequence diverges.

Recursive Sequences

Each term in the sequence is defined based on previous terms, not from a given formula.

$a_1 = 1 \qquad a_n = 3a_{n-1} \qquad n \geq 2$

→ No way of getting a_5 directly.

→ Need to step through one-by-one.

$a_1 = 1 \qquad a_2 = 3 \cdot a_1 = 3 \qquad a_3 = 3 \cdot a_2 = 9$

$a_4 = 3 \cdot a_3 = 27 \qquad a_5 = 3 \cdot a_4 = 81$

Example: Find the first 5 terms of the sequence defined by $a_1 = 2$, $a_2 = 3$ and $a_{n+2} = 2a_{n+1} - a_n$ for $n \geq 1$.

$$a_3 = 2a_2 - a_1 = 2\cdot(3) - 2 = \underline{4}$$

$$a_4 = 2a_3 - a_2 = 2(4) - 3 = \underline{5}$$

$$a_5 = 2a_4 - a_3 = 2(5) - 4 = \underline{6}$$

3 Limit Laws and Theorems for Sequences

All of the limit laws work the same way that they did before. We also have a version of the squeeze theorem.

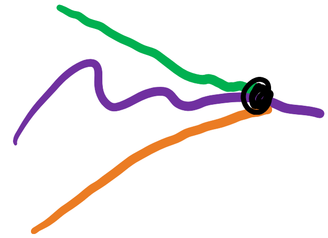
Thm Assume $\{a_n\}$ and $\{b_n\}$ are convergent sequences with

$$\lim_{n\to\infty} a_n = L \quad \text{and} \quad \lim_{n\to\infty} b_n = M$$

Then

- $\lim_{n\to\infty} a_n + b_n = L + M$
- $\lim_{n\to\infty} a_n b_n = LM$
- $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{L}{M}$ provided $M \neq 0$
- $\lim_{n\to\infty} c\, a_n = cL$ c a real number.

Squeeze Theorem



If I have sequences $\{l_n\}$, $\{a_n\}$ $\{u_n\}$ so that

$$l_n \le a_n \le u_n \quad \text{and}$$

$$\lim_{n\to\infty} l_n = L = \lim_{n\to\infty} u_n$$

Then $\lim_{n\to\infty} a_n$ exists and equals L.

Example: Find

$$\lim_{n\to\infty} \overset{\nearrow 0}{\sin(n)e^{-n^2}} + \overset{\nearrow 9}{\left(3-\frac{1}{n}\right)^2} = \boxed{9}$$

$$\lim_{n\to\infty} \sin(n)e^{-n^2} + \underbrace{\lim_{n\to\infty}\left(3-\frac{1}{n}\right)^2}$$

$\lim_{n\to\infty} \sin(n)$ does not exist

$$\lim_{x\to\infty}\left(3-\frac{1}{x}\right)^2 = 9$$

$$-e^{-n^2} \le \sin(n)e^{-n^2} \le e^{-n^2}$$

$$\downarrow \qquad\qquad \downarrow$$

$$0 \qquad\qquad 0$$

By the squeeze theorem $\lim_{n\to\infty} \sin(n)e^{-n^2} = 0$ exists!

4 Functions of Sequences

You can also apply functions to sequences, and that all works in the limit too, provided the function is continuous.

$\{a_n\}$ $a_n = n^2 - 3n$

$f(x) = \sin(x)$ $f(a_n) = \sin(n^2 - 3n)$

New sequence $\{f(a_n)\}$

Statement If f is continuous and $\lim_{n\to\infty} a_n = L$ then

$$\lim_{n\to\infty} f(a_n) = f(L)$$

$$f(L) = f\left(\lim_{n\to\infty} a_n\right)$$

Example: For $f(x) = e^x$ and $a_n = \frac{\sin n}{n^2}$, what is

$$\lim_{n\to\infty} f(a_n)?$$

What is $\lim_{n\to\infty} e^{\frac{\sin(n)}{n^2}}$?

This is $f(L) = e^L$ for $L = \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\sin(n)}{n^2}$

Because $-\frac{1}{n^2} \le \frac{\sin(n)}{n^2} \le \frac{1}{n^2}$

$\downarrow \qquad \downarrow \qquad \downarrow$ Squeeze.

$0 \qquad 0 \qquad 0$

Thus $L = 0$, so the limit we want is $e^0 = \boxed{1}$

5 Bounded Sequences

- Terms don't get too big or too small.

Def. A sequence $\{a_n\}$ has a lower bound M if $M \le a_n$ for all n.

→ Floor for the values of a_n

- A sequence $\{a_n\}$ has an upper bound K if $a_n \le K$ for all n.

→ Ceiling for the values of a_n.

- We say $\{a_n\}$ is:
  - bounded from below if it has a lower bound
  - bounded from above if it has an upper bound
  - bounded if it has both.

Ex

$a_n = n^2 \quad n \geq 1$

- Bounded from below $\quad a_n \geq 0$ all n
- Not bounded from above

$b_n = 1/n \quad n \geq 1$ $\qquad b_n \geq 0$

$b_n \leq 2$

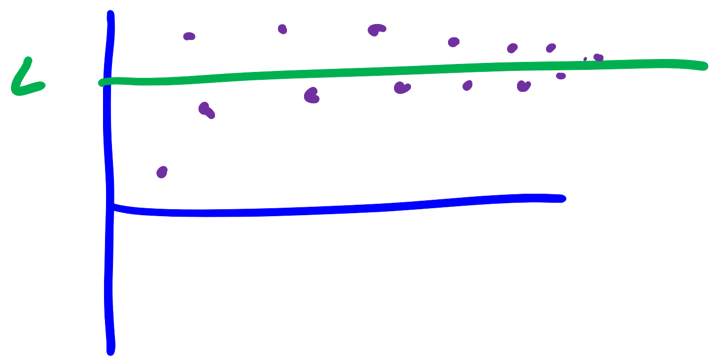
- Bounded

Def If a sequence is not bounded (both sides) we say it is unbounded.

Theorems

Theorem 5:

Any convergent sequence is bounded.



Theorem 6:

Any bounded, monotone sequence converges.

Increasing: $a_{n+1} \geq a_n$ for all n

Decreasing: $a_{n+1} \leq a_n$ for all n.

Monotone: Either increasing or decreasing.