Complex Numbers Part 2

Learning Goals

- Compute powers of complex numbers using the exponential form
- Compute all of the nth roots of a complex number
- Use De Moivre's Formula to find formulas for $\cos(n\theta)$ and $\sin(n\theta)$.
- Find the nth roots of unity.
- Find the complex roots of polynomials.

Contents

1 Powers of Complex Numbers

Multiplying complex numbers is really easy in polar form.

What does this mean for taking powers of complex numbers?

What about negative powers of complex numbers?

Geometry of Powers

Example: Compute z^{-1}, z^2, z^3, and z^4 for the complex number $z = 1 - \sqrt{3}i$.

2 Roots of Complex Numbers

Take a complex number $w = re^{i\theta}$, and say that we want to solve the equation $z^n = w$. How could we do this?

What other values of z are there?

General Formulas

For the complex number $w = re^{i\theta}$, and a positive number n, the solutions to the equation $z^n = w$ are

Example: Find the polar coordinates of the 3 complex solutions of $z^3 = 1+i$.

3 Roots of Unity

An important result of roots of complex numbers is the idea of roots of unity.

Definition. The **nth roots of unity** are the n complex solutions to the equation $z^n = 1$.

What does this look like plotted out?

Example: Find the polar and rectangular coordinates of the 6th roots of unity.

4 De Moivre's Formulas

We have easy formulas for the powers of complex numbers. What else does that give us?

De Moivre's Formulas

For any θ and any positive integer n, we have that

Example: Use De Moivre's Formulas to derive the double angle formulas for sine and cosine.

5 Polynomials and Complex Numbers

The main reason we care about complex numbers is the Fundamental Theorem of Algebra:

Theorem. *Any polynomial equation of the form* $P(z) = 0$ *of degree* $n \geq 1$ *has exactly* n *complex solutions, counting repeated roots.*

What does this mean?

What other facts do we have?

Example: Find all solutions of the equation $x^5 + z^3 - z^2 - 1$.