Complex Numbers

Learning Goals

- Add, subtract, multiply, and divide complex numbers
- Find the absolute value of a complex number
- Convert complex numbers to and from polar form
- Find the product and quotient of complex numbers in polar form
- Use complex numbers to help solve partial fraction problems
- Use complex numbers to discuss the radius of convergence of power series

Contents

1 Algebra of Complex Numbers

Why Complex Numbers?

We know that the polynomial $x^2 + 1$ has no real roots. But what if it had roots?

Definition. A **complex number** z is defined as

$$z = x + iy$$

for x and y real numbers. For this number x is the **real part** of z and y is the **imaginary part** of z.

Operations on Complex Numbers

All of the operations we can do on real numbers we can also do with complex numbers. The idea is that we treat i like a variable and group terms so it matches the form of a complex number.

Example: Compute the following:

1. $(2 + 3i) - (5 - 4i)$
2. $3(2 + i) + 4(4 - i)$
3. $(2 + i)(3 - 2i)$

2 Complex Conjugate and Division

The last operation on real numbers that we want to extend to complex numbers is division. How do we think about division of real numbers?

For this, we need another definition:

Definition. The **complex conjugate** of $z = x + iy$ is the complex number $\bar{z} = x - iy$. The **modulus** of a complex number $z = x + iy$ is $|z| = \sqrt{x^2 + y^2}$.

Properties of Complex Conjugates

(a) $\bar{\bar{z}}$

(b) $\overline{z_1 + z_2}$

(c) $\overline{z_1 \cdot z_2}$

(d) $z \cdot \bar{z}$

Reciprocal of Complex Numbers

If we look at the product $z \cdot \dfrac{\bar{z}}{|z|^2}$, what do we get?

Example: Find $\dfrac{2+3i}{1-i}$.

3 Geometry of Complex Numbers

How can we visualize complex numbers? The notation $z = x+iy$ is suggestive here, in that we can use the x and y coordinates in the plane to plot and view complex numbers.

We can also use polar coordinates to view these numbers.

Example: Plot the complex number $2 - i$ in the complex plane as well as the polar coordinates of this number.

4 Exponential Form and Euler's Formula

When we write a complex number in polar form, we see that it can be written as

$$z = |z| \cos \theta + i|z| \sin \theta = |z|(\cos \theta + i \sin \theta)$$

Is there a nicer way to view this?

Definition. The **polar form** of a complex number z is $|z|e^{i\theta}$.

Product and Quotient in Polar Form

It is really easy to add and subtract complex numbers in rectangular form $x+iy$, and slightly more complicated to multiply and divide in this form. For polar form or exponential form, however, multiplying and dividing is really easy.

Example: Convert to exponential form and then find the quotient $\frac{3-3i}{1+\sqrt{3}i}$.

5 Applications to Partial Fractions

We want to see how complex numbers can be used to help with some Calculus topics. The first concept is partial fractions. What was the issue with handling irreducible polynomials before?

Method:

Example: Use complex numbers to help compute $\displaystyle\int \frac{12x}{(x+1)(x^2+2x+5)}\,dx$

6 Applications to Power Series

Complex numbers are also useful for interpreting power series and the radius of convergence.

This can also be tied to power series expansions.

Conclusion:

If there is a place where the function doesn't exist in the **complex plane**, that gives me a point where the series can't converge, and so an upper bound on the radius.

Example: Show that the function $f(x) = \frac{x}{x^4+64}$ is undefined at the four complex numbers given by $\pm 2 \pm 2i$. Use this fact to show that the radius of convergence of the power series for $f(x)$ centered at zero is no more than $2\sqrt{2}$. Find the actual power series and validate this.

Example: Use complex numbers to help find $\displaystyle\int \frac{13}{(x-2)^2(x^2+9)}\,dx$

Example: Use complex numbers to find an upper bound for the radius of convergence of the power series expansion of $\frac{1}{(x^2-4x+5)^2}$ centered at $x = -1$.