Method of Partial Fractions

Learning Goals

- Find the partial fraction decomposition of a given rational function
- Integrate a rational function by first using long division and then the method of partial fractions
- Integrate a rational function with linear and/or irreducible quadratic factors with multiplicity 1 using the method of partial fractions
- Integrate a rational function with repeated linear and/or irreducible quadratic factors using the method of partial fractions

Contents

1 Partial Fraction Decompositions

In this section, we have one more technique for doing integrals. We'll start by setting this up and then see how it helps with integrals.

What is a Partial Fraction Decomposition?

What do we know?

If $\dfrac{p(x)}{q(x)}$ is a rational function with degree of $p(x)$ less than the degree of $q(x)$, and $q(x)$ can be written as

$$q(x) = (x - a_1)(x - a_2) \cdots (x - a_k)$$

Example: Find the Partial Fraction Decomposition for

$$\frac{2x^2 + 5x - 12}{x(x-4)(x+1)}$$

2 Value Substitution and Integrals

For more complicated polynomials, it can be difficult to solve for the necessary coefficients by this method. However, we have another trick we can use.

Example: Compute $\displaystyle\int \frac{x^2+3}{(x+1)(x+2)(x-4)}$

3 Irreducible Quadratics

The method as described previously works when the integrand is a rational function with the following properties:

- The denominator can be completely factored into linear factors
- No linear factor is repeated in this factorization
- The degree of the numerator is less than the degree of the denominator

We will now deal with each of the conditions above, so that we end up with a method that works for all rational functions.

Not all linear factors

Not all polynomials can be completely factored into linear factors.

Handling Quadratics

In order to get the right number of coefficients to make these systems work, we need to have both an x term and a constant term on top of the irreducible quadratic.

Example: Compute $\displaystyle\int \frac{3x+4}{(x-1)(x^2+9)}$

4 Repeated Factors

What happens if a factor is repeated in the denominator?

Example: Compute $\displaystyle\int \frac{2x}{(x+1)^2(x-3)}$

5 Long Division

What if the degree on top is higher than the bottom? We can't solve it in the normal way, because we don't have enough information.

Example: Compute $\displaystyle\int \frac{x^3+2x+1}{x^2-1}\,dx$

6 Combining all the Adjustments

Some problems need more than one of these adjustments, and also add in completing a square.

Example: Compute $\displaystyle\int \frac{25}{x(x^2+2x+5)^2}\,dx$