

Area between Two Curves

Learning Goals

- Compute the area of vertically simple regions in the plane
- Compute the area of horizontally simple regions in the plane
- Recognize regions as vertically simple or horizontally simple, or split regions into ones that have this property
- Compute the area bounded by two curves
- Set up and compute areas of regions where multiple integrals are required

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1 Introduction

Area between two curves is a first application of integration, of which there will be many more. The general idea is finding areas of regions in the plane, some of which are easier than others.

Easiest Type of Region

The definition of an integral gives us an easy way to find the area of very specific regions in the plane.

We can do something a little more general as well.

Regions like this are called **vertically simple**.

Example: Find the area between the graphs of $y = x + 1$ and $y = x^2 + 3x + 2$ over the interval $[1, 4]$.

2 Region Bounded by Two Curves

Another type of question that can be answered by these methods is finding the area of the region bounded between two curves.

Example: Find the area of the region bounded by the graphs of $y = x^2 + 2$ and $y = 3x + 6$.

3 Curves that Cross Multiple Times

There can also be situations where the area of a different type of region is needed. This can arise from either an interval being given, and the two curves crossing in the middle, or the area bounded by two curves that cross multiple times.

Example: Find the area between the graphs of $y = x^2 - 4$ and $y = 3x - 4$ over the interval $[1, 5]$.

4 Multiple Vertically Simple Regions

Sometimes, it is not so easy to find the area of a region using an integral. The reason the regions discussed previously were so easy is because there was one function on the top and one function on the bottom over the entire region. That is not always the case.

Process for these Problems

For pretty much all cases, it is possible to split the region into different pieces that are vertically simple. This will require several different integrals to find the total area.

Example: Find the area bounded between the graphs of $y = 2x$, $y = 5x$, and $y = -x + 12$

5 Horizontally Simple Regions

The idea for finding area that has been developed so far has been ‘adding up’ (integrating) these small vertical lines to get the area under the curve. There’s nothing saying this can’t be done for horizontal lines as well.

Definition: We say that a region is *horizontally simple* if there is a one curve on the left-side of the region and one curve on the right-side of the region over all of it.

For a region like this, if we let $x = g(y)$ be the ‘right’ curve and $x = f(y)$ be the ‘left’ curve, then we can find the area of the region between $g(y)$ and $f(y)$ as

Ways this can be used

1. Regions where the boundaries are given as x equals a function of y
2. Regions that are horizontally simple and it is possible to solve out for x as a function of y .
3. Regions where it is easier to visualize the region and add it up using horizontal lines as opposed to vertical lines.

All of the methods discussed previously (multiple regions, curves crossing) can also be done for horizontally simple regions, or regions where you want to use y integrals.

Example: Find the area bounded between the graphs of $y = 2x$, $y = 5x$, and $y = -x + 12$ using integrals in y .