Arc Length and Area in Polar Coordinates

Learning Goals

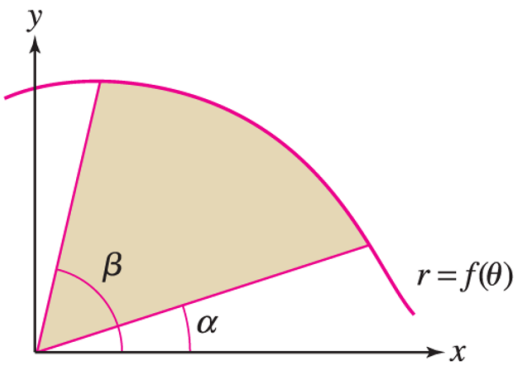
- Find the area of a region bounded by a polar curve
- Find the area of a region between two polar curves
- Find the arc length of a polar curve

Contents

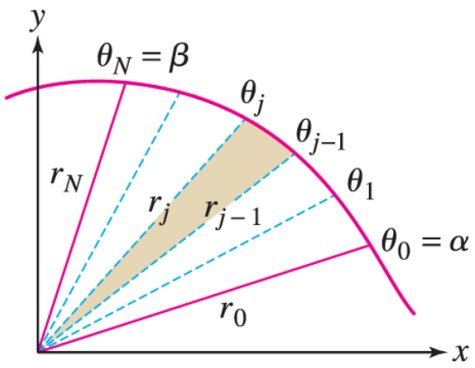
1 Area for Polar Functions

Assume that we write $r = f(\theta)$ with $f(\theta) > 0$ for $\alpha \leq \theta \leq \beta$. We want to find the area enclosed inside the graph and the sector between $\theta = \alpha$ and $\theta = \beta$.

How do we find this area here?



(A) Region $\alpha \le \theta \le \beta$



(B) Region divided into narrow sectors

Rogawski et al., *Calculus: Early Transcendentals*, 4e, © 2019 W. H. Freeman and Company

We want to add up all of these little triangles/sectors.

Theorem. *If f is a continuous function with $f \geq 0$ then the area bounded by a curve in polar form $r = f(\theta)$ and the rays $\theta = \alpha$ and $\theta = \beta$ is given by*

How does this give area?

Example: Find the area of the portion of the circle $r = 2\cos\theta$ between the rays $\theta = -\pi/4$ and $\theta = \pi/4$.

2 More Examples

Example: Find the area of one petal of the graph $r = \sin 5\theta$

Example: Find the area inside the circle $r = 4\cos\theta$ and outside the circle $r = 2$.

3 Arc Length

Let's now try to figure out a new arc length formula in polar coordinates. What was our parametric formula?

How can we get a formula in polar coordinates?

Algebra

Theorem. *Let $f'(\theta)$ be continuous on $[\alpha, \beta]$. Then the arc length s of the curve $r = f(\theta)$ is given by*

Example: Find the length of one petal the curve $r = \sin(5\theta)$.