Power Series

Learning Goals

- Identify a power series
- Find the interval and radius of convergence for a power series
- Add and multiply two power series together
- Find the power series representation of a function using a known power series
- Find the function represented by a given power series
- Differentiate and integrate a power series

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1 Definition of Power Series

This section leads into power series, which is a very important tool in a lot of physical applications. A lot of well-known functions can be written as power series, and certain functions, like Bessel functions (which are very common in physics applications), can *only* be written as power series.

Definition: A **power series** with *center* c is an infinite series of the form

$$F(x) = \sum_{n=0}^{\infty} a_n (x - c)^n.$$

Operations on Power Series

Power Series can formally be treated like polynomials, but care is needed at each step.

Take two power series centered at zero

$$F(x) = \sum_{n=0}^{\infty} a_n x^n \qquad G(x) = \sum_{n=0}^{\infty} b_n x^n$$

2 Convergence of Power Series

Power Series are a type of infinite series, so we need to talk about convergence of this series. However, now convergence will depend on the value of x.

Example: For what values of x does the series $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{3^n}$ converge?

It turns out the type of answer we got for the previous example is not a coincidence. The fact that this was an interval is *always* what happens.

Definition: For any power series,

$$F(x) = \sum_{n=0}^{\infty} a_n (x - c)^n.$$

there is an interval of x values on which it converges. This is called the **interval of convergence** for that power series, and is an interval centered around c.

Radius of Convergence

$$F(x) = \sum_{n=0}^{\infty} a_n (x - c)^n.$$

Types of Convergence

We know the series converges on this interval. Is more always true?

Example: Where does the power series $\sum_{n=1}^{\infty} \frac{x^n}{n}$ converge?

3 Power Series Expansions

We have these power series. They may converge, but we might not know what they actually converge to (as functions). If we know what the function that we get as a result is, then we say that this power series is a **power series expansion** of that function.

Example: What do we know about the power series $\sum_{n=0}^{\infty} x^n$?

How can we find power series expansions?

Example: Find a power series expansion for the function $\dfrac{1}{1+2x^3}$. Where is this expansion valid?

4 Differentiation and Integration of Power Series

The main reason power series are so useful is because of the following properties:

Theorem. *Assume that the power series* $F(x) = \sum_{n=0}^{\infty} a_n(x-c)^n$ *has radius of convergence* $R > 0$ *(or* ∞*). Then*

Example: Find a power series expansion for $\arctan x$. Where is this expansion valid?

5 A Famous Example

Let's consider a new power series:

$$F(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

What do we know about this power series?

Example: Find a power series expansion for

$$f(x) = \frac{1}{x^2 - 4x + 4}.$$