# Ratio and Root Tests

## Learning Goals

- Determine if a series converges or diverges using the ratio test
- Determine if a series converges or diverges using the root test
- Choose an appropriate convergence test for a series
- Determine if a series converges or diverges using any method/test

## Contents

# 1 Ratio Test

This section covers two more tests for evaluating whether or not a series converges or diverges. They work for series with both positive and negative terms, but sort of ignore that fact by taking absolute values first. The extra benefit they have is that they do not require the series to alternate in order to give a result.

**Theorem** (Ratio Test). *Let $\sum a_n$ be a series that we want to analyze. Assume that the following limit exists*

$$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

*Then*

The idea as to why this works is direct comparison to a geometric series, which we will illustrate later.

**Example:** Does $\sum_{n=1}^{\infty} \frac{n^2}{3^n}$ converge?

## 2 Root Test

The other test we have in this section is the Root Test. It does the same thing, but with roots instead of ratios.

**Theorem** (Root Test)**.** *Let* $\sum a_n$ *be a series that we want to analyze. Assume that the following limit exists*

$$t = \lim_{n \to \infty} \sqrt[n]{|a_n|}$$

*Then*

The idea of the proof is the same, but it is more complicated.

**Example:** Does $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2n}{n+4}\right)^n$ converge or diverge?

# 3 Proof of Ratio Test

**Example:** Does $\sum_{n=2}^{\infty} \frac{5^n}{n!}$ converge or diverge?

# 4 Choosing Tests

How do we choose which test to use in a given case? Which is the best order to attempt these tests to make the process as simple as possible?

**First**, try the $n$th term divergence test. Remember this can **only** tell you that a series diverges, not that it converges.

If a series does not have all positive terms, you have basically two options:

If a series has positive terms (or you made it that way by taking absolute values) now we have more options.

(a) Direct Comparison Test

(b) Limit Comparison test

(c) Ratio Test

(d) Root Test

(e) Integral Test

**Examples:** Analyze each of the following series and determine whether they converge or diverge.

$$\sum_{n=1}^{\infty} \frac{1}{(n+2)^n}$$

$$\sum_{n=2}^{\infty} \frac{1}{n^2 - \sqrt{n}}$$

$$\sum_{n=1}^{\infty} \frac{n!}{(2n)!}$$

$$\sum_{n=1}^{\infty} \frac{3^n}{n!}$$